Viscous Relativistic Hydrodynamics in Heavy Ion Collisions

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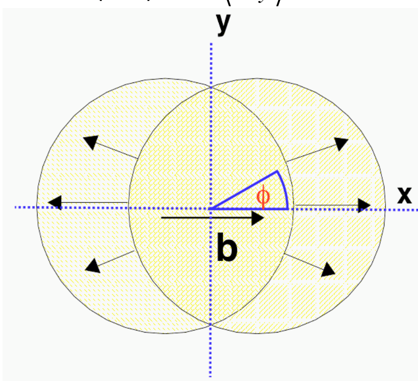
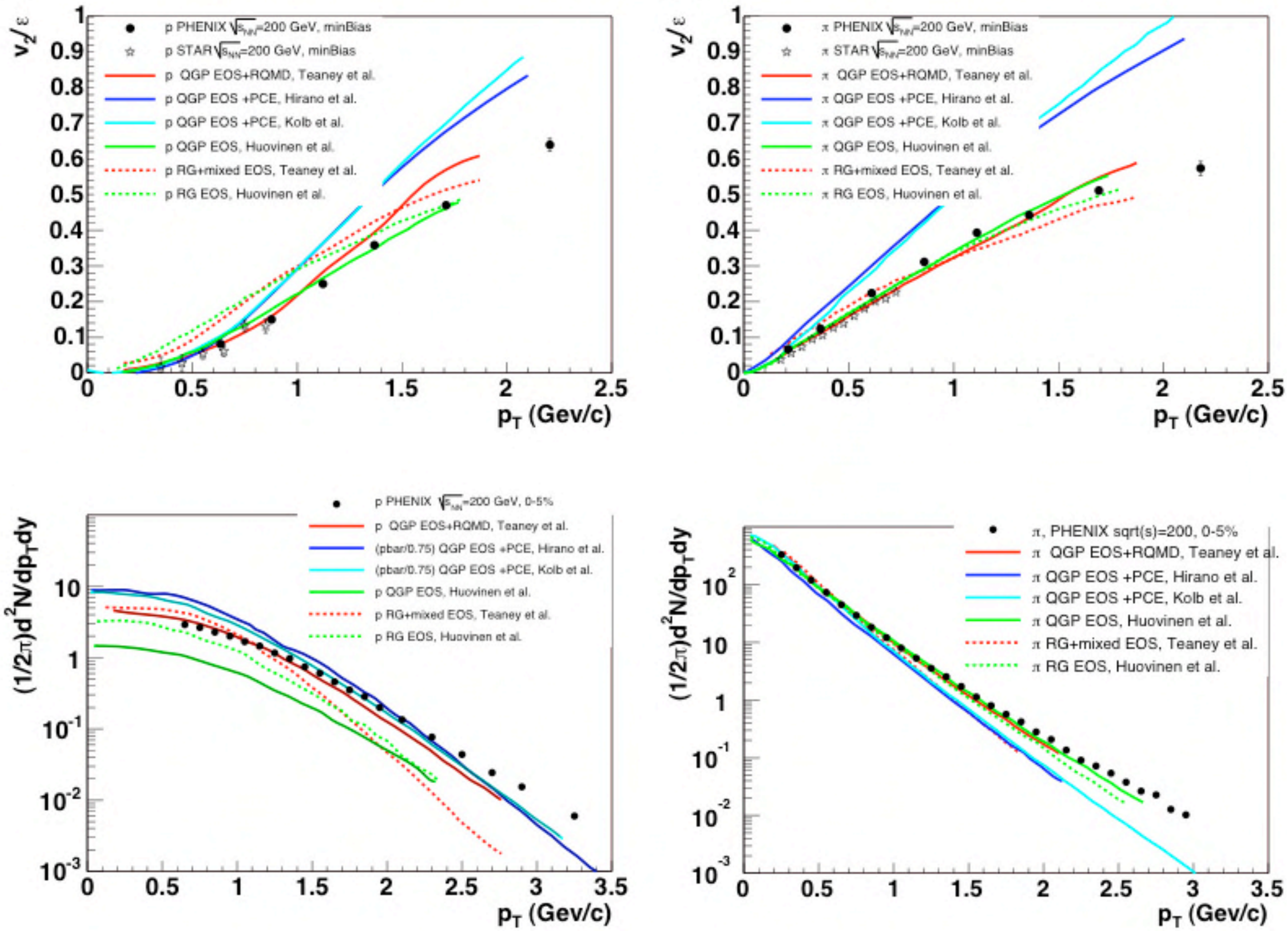
Contents:

Success of Ideal Hydro:

$$v_2 \equiv \frac{\langle p_x \rangle^2 - \langle p_y \rangle^2}{\langle p_x \rangle^2 + \langle p_y \rangle^2} \approx 20\%$$



Proton **Pion**



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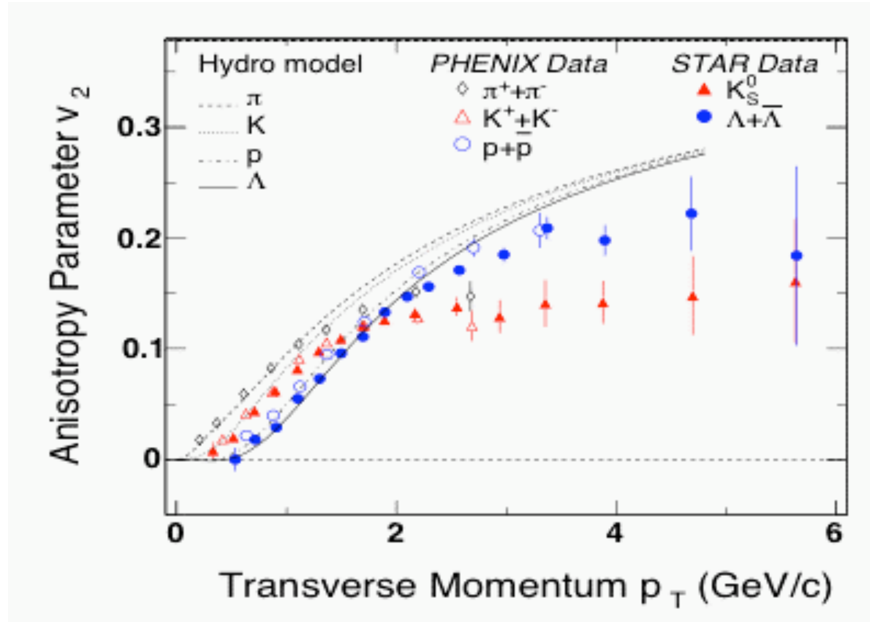
Breakdown of Hydro:

- Deviations from hydro seen at:
  - Large p_T
  - Peripheral collisions
  - Collisions at lower energies
  - Forward rapidities



- **V_2 reflects QGP expansion state and partly on phase transition**

- **V_2 key to understanding QGP viscosity -- need 2D simulations**

Viscous Corrections:

$$\tau = \sqrt{t^2 - z^2}$$

$$\eta_s = \frac{1}{2}\ln\left(\frac{t+z}{t-z}\right)$$

- Consider viscous boost invariant expansion:

$$\frac{d(\tau s)}{d\tau} = \begin{cases} \dfrac{\frac{4}{3}\eta}{\tau T} & \text{(viscous)} \\ 0 & \text{(ideal)} \end{cases}$$

- Validity of Ideal hydrodynamics: $\dfrac{\delta T_{\mu\nu}}{T_{\mu\nu}} << 1 \longrightarrow \dfrac{\frac{4}{3}\eta}{\tau T}\tau << \tau s$

- Define sound attenuation length: $\Gamma_s \equiv \dfrac{\frac{4}{3}\eta}{sT}$

$$\boxed{\frac{\Gamma_s}{\tau} << 1}$$

- **For ideal hydro to work mean free path should less than expansion rate**

Estimates of η/s in QGP:

- **pQGP - Kinetic Theory** **(M. Le Bellac)**

$$\left.\begin{aligned} \frac{1}{\eta} \approx \frac{1}{T^3}\alpha_s^2 \ln\left(\frac{1}{\alpha_s}\right) &\xrightarrow{\alpha_x = 0.5} \eta \approx 5.8T^3 \\ s_{QGP} &= 37\frac{4\pi^2}{90}T^3 \end{aligned}\right\} \left(\frac{\Gamma}{\tau}\right)_{pert} \approx 0.36\frac{1}{\tau T}$$

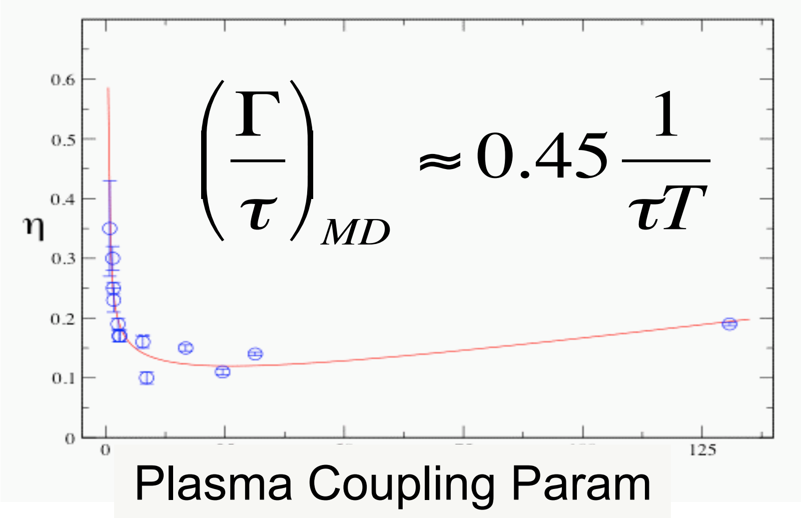
- **N=4 SUSY YM** **(Policastro, Son, Starinets) (Gubser, Klebanov, Tseytlin)**

$$\left.\begin{aligned} \eta &= \frac{\pi}{8}N_c^2 T^3 \\ s &= \frac{\pi^2}{2}N_c^2 T^3 \end{aligned}\right\} \left(\frac{\Gamma}{\tau}\right)_{AdS/CFT} \approx 0.1\frac{1}{\tau T}$$

Estimates of η/s in QGP:

- **MD Simulations of cQGP**

  (Gelman, Shuryak, Zahed)

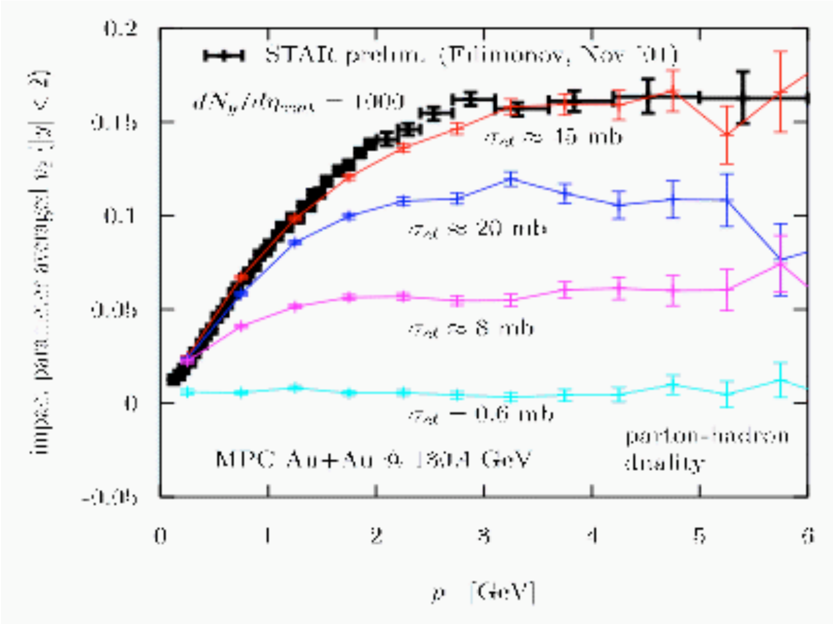


- **Phenomenology**

  (Gyulassy, Molnar)

$$\eta \approx 1.264\,\frac{T}{\sigma_0}$$

$$\left(\frac{\Gamma}{\tau}\right)_{GM} \approx 0.03\,\frac{1}{\tau T}$$



Evolution:

- Value of viscosity uncertain: $\frac{\eta}{s} = 0.2$

- At time τ_0

$$T_0 \approx 300 \text{ MeV and } \tau_0 \approx 1 \text{ fm}$$

$$\frac{\Gamma}{\tau} \approx 0.175$$

- How does Γ/τ evolve?

    - Look at Bjorken expansion...

Evolution:

- 1D Bjorken Expansion: $T \propto \frac{1}{\tau^{1/3}}$
  - Scale Invariant Cross section: $\sigma \sim \frac{\alpha_S^2}{T^2}$

$$\frac{\Gamma_s}{\tau} \sim \frac{1}{\tau T} \sim \frac{1}{\tau^{2/3}}$$

Rapid Thermalization

  - Constant Cross section: $\sigma \sim const.$ $\tau n \propto const$

$$\frac{\Gamma_s}{\tau} \sim \frac{l_{mfp}}{\tau} \sim \frac{1}{\tau n \sigma} \sim 1$$

Const Thermalization

  - System more likely to thermalize when $\eta \propto T^3$
  - Large opacity needed in GM an artifact of fixed scale

Relativistic Navier Stokes Equations (RNSE)

- RNSE difficult to solve
- unstable modes propagate faster then speed of light
- Cannot predict future evolution of initial fluid states (Hiscock, Lindbolm)

- RNS stress tensor changes instantly

$$T_{vis}^{ij}\Big|_{\text{instantly}} = \eta\left(\partial^i v^j + \partial^j v^i - \frac{2}{3}\delta^{ij}\partial_i v^i\right)$$

- There are a number of models which relax to RNSE

$$T_{vis}^{ij}\Big|_{\omega\to 0} \sim \eta\left(\partial^i v^j + \partial^j v^i - \frac{2}{3}\delta^{ij}\partial_i v^i\right)$$

- When hydrodynamics is valid these models all agree with RNSE
  - (L. Lindbolm)

Typical Relaxation Process: Diffusion

- Violates causality
- Breaks sum rules associated with current conservation

- Continuity: $\partial_t n + \partial_x j = 0$

$$\partial_t n - D\nabla^2 n = 0$$

- Ficks Law: $\partial_t j = -D\nabla n$

- Relaxation Time Approximation: $\partial_t j = -\dfrac{(j + D\nabla n)}{\tau_R}$

$$\tau_R \partial_t^2 n + \partial_t n - D\nabla^2 n = 0 \qquad \lambda = \pm\sqrt{\frac{D}{\tau_R}}$$

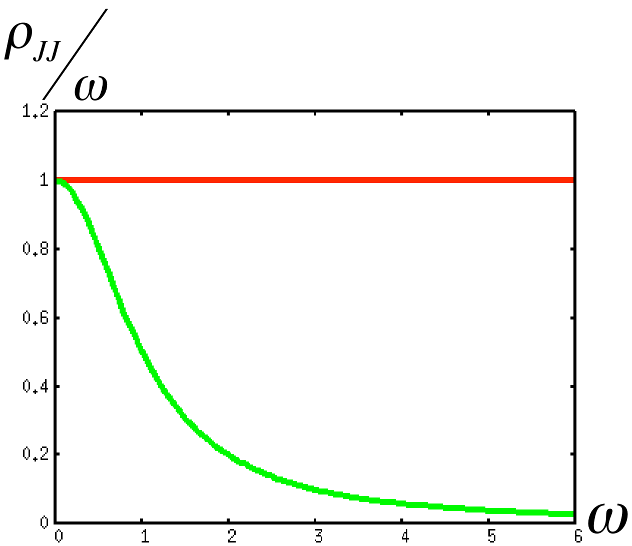
Typical Relaxation Process: Diffusion

- Spectral Density from ordinary Diffusion Equation

$$\frac{\pi\rho_{JJ}(\vec{k}=0,\omega)}{\omega}=\chi_S D$$

- Spectral Density in Relaxation Time Approx.

$$\frac{\pi\rho_{JJ}(\vec{k}=0,\omega)}{\omega}=\frac{\chi_S D}{1+\left(\omega\tau_R\right)^2}$$



$$\int\frac{\rho_{JJ}(\vec{k}=0,\omega)}{\omega}d\omega=\chi_S\left\langle v_{th}^2\right\rangle \qquad \text{(Weak Coupling)}$$

- Have short and long time parameters
  - Long Time: *D*
  - Short Time: $\frac{D}{\tau_R}=\left\langle v_{th}^2\right\rangle$

In General:

$$\vec{j} = -L\vec{F} \Longrightarrow \vec{j} = -L\vec{F} - \tau_R \partial_t \vec{j}$$

$$t_m << \partial_t j \qquad\qquad t_m \sim \partial_t j$$

Analogous set of equations for viscous hydrodynamics:

$$\tau_R \frac{d\Pi}{d\tau} = -\varsigma \partial_\alpha u^\alpha - \Pi \qquad \tau_R \frac{d\pi^{\mu\nu}}{d\tau} = \eta \langle \partial^\mu u^\nu \rangle - \pi^{\mu\nu} \qquad \tau_R \frac{d\nu^\mu}{d\tau} = -\kappa \partial_\mu \left(\frac{\mu}{T} \right) - \nu^\mu$$

ζ: bulk viscosity
η: shear viscosity
κ: thermal conductivity

This is second order (Truncated) Israel-Stewart:

$$T^{\mu\upsilon} = \varepsilon u^\mu u^\nu - (p + \Pi)\Delta^{\mu\nu} + \pi^{\langle\mu\nu\rangle}$$

$$N^\mu = n u^\mu + \Delta^{\mu\nu} v_\nu$$

Same Hydro Equations:

$$\partial_\mu T^{\mu\nu} = 0$$

$$\partial_\mu N^\mu = 0$$

GENERIC Formalism:

- Same principal as IS
- For Details: H. C. Öttinger, Physica A 254 (1998) 433.
- Why use GENERIC Structure?
  - Numerically easier to implement
  - Not necessarily restricted to small deviations from equilibrium
- Introduce tensor which evolves in time as:

$$u^{\lambda}\left(\partial_{\lambda} c_{\mu\nu} - \partial_{\mu} c_{\lambda\nu} - \partial_{\nu} c_{\mu\lambda}\right) = -\frac{1}{c\tau_0}\bar{c}^{\mu\nu} - \frac{1}{c\tau_2}\left\langle c^{\mu\nu}\right\rangle \qquad \bar{c}_{ij} = \left(tr\ c\right)\delta_{ij} \ \text{ and } \ \left\langle c_{ij}\right\rangle = c_{ij} - \frac{1}{3}\bar{c}_{ij}$$

- This leads to its rapid relaxation to velocity gradients:

$$\partial_t c_{ij} - \left(\partial_i v_j + \partial_j v_i\right) = -\frac{\bar{c}_{ij}}{\tau_0} - \frac{\left\langle c_{ij}\right\rangle}{\tau_2}$$

- For small relaxation times:

$$c^{ij} \approx \tau_0 \frac{2}{3}\delta^{ij}\partial_k v^k + \tau_2\left(\partial^i v^j + \partial^j v^i - \frac{2}{3}\delta^{ij}\partial_k v^k\right)$$

Generic Formalism:

- The energy-momentum tensor in the LRF is given by:

$$T^{ij}_{LRF} = p_0\delta^{ij} - \phi\left[c^{ij} - \left(c^{ik}c_k^{\ j}\right)\right] \qquad \phi = 4\frac{\partial\varepsilon}{\partial tr\ \mathrm{c}^2} \geq 0$$

- Assuming a specific thermodynamic relation of the form:

$$\varepsilon = \varepsilon_0 + \frac{1}{2}\alpha\ \mathrm{tr}\ c^2$$

- For small deviations from equilibrium:

$$T^{ij}_{LRF} = p_0\left(\delta^{ij} - \frac{2\alpha}{p_0}c^{ij}\right) \longrightarrow \eta = 2\tau_2\alpha$$

- Limit c for larger deviations from equilibrium:

$$T^{ij}_{LRF} = p_0\left(\delta^{ij} - \frac{2\alpha}{p_0}\frac{c^{ij}}{\sqrt{1+c^2}}\right)$$

Solution to Relaxation Time Equations:

- Bjorken Expansion: 1st order viscous hydro

$$\frac{d\varepsilon}{d\tau} = -\frac{\varepsilon + T^{zz}_{eq}}{\tau} \qquad T^{zz}_{eq} = p - \frac{4}{3}\frac{\eta}{\tau}$$

- Bjorken Expansion: Relaxation time approximation

$$\frac{d\varepsilon}{d\tau} = -\frac{\varepsilon + T^{zz}}{\tau} \qquad \frac{dT^{zz}}{d\tau} = -\frac{T^{zz} - T^{zz}_{eq}}{\tau_R}$$

- In any relaxation scheme have two parameters and IC:

$$\frac{\eta}{s} = 0.2 \qquad \tau_R = 2\eta\frac{3}{4p} \qquad T^{zz}_0 = T^{zz}_{eq}$$

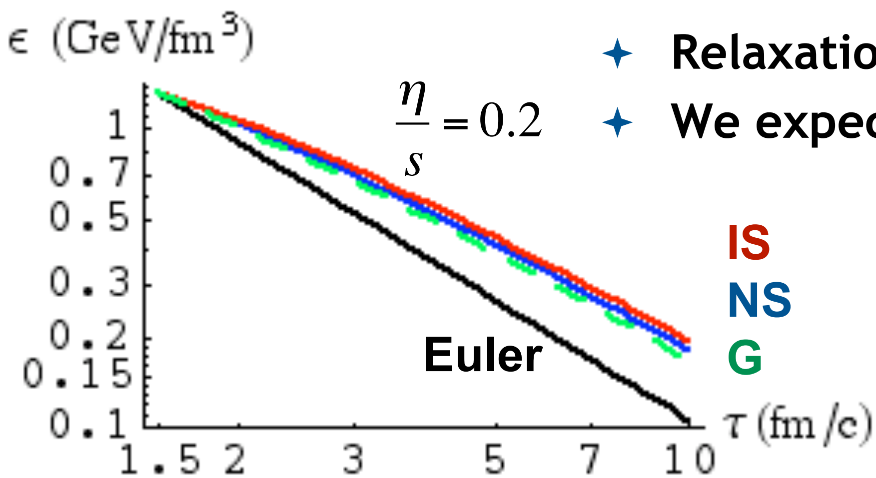
Solution to Relaxation Time Equations:

- Bjorken expansion with no transverse flow

  - NS: $$\frac{d\varepsilon}{d\tau} + \frac{1}{\tau}(\varepsilon + p) = \frac{4}{3}\frac{\eta}{\tau^2}$$

  - IS: $$\frac{d\varepsilon}{d\tau} + \frac{1}{\tau}(\varepsilon + p) = \frac{\pi^{zz}}{\tau} \qquad \tau_R \frac{d\pi^{zz}}{d\tau} = \frac{4}{3}\frac{\eta}{\tau} - \pi^{zz}$$

  - GENERIC: $$\frac{d\varepsilon}{d\tau} + \frac{1}{\tau}\varepsilon = -\frac{p}{\tau}(1 - a_1 c_{33}) \qquad \frac{\partial c_{33}}{\partial \tau} - \frac{2}{\tau}(c_{33} - 1) = \frac{1}{\tau_2}\left(c_{33} - \frac{1}{3} tr\, c\right)$$



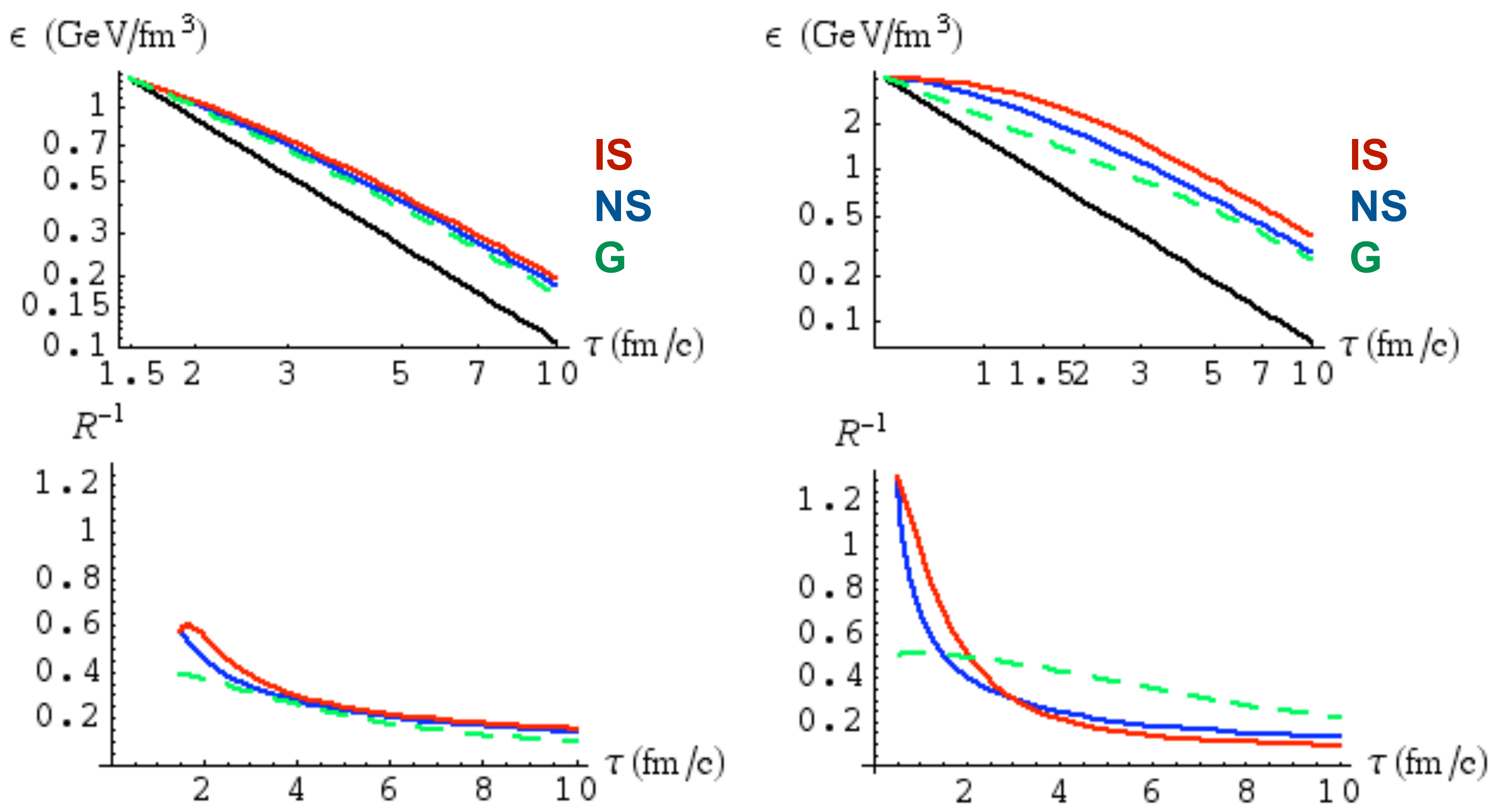
- **Relaxation Methods agree with NS**
- **We expect disagreement far from equilibrium**

$$R^{-1} = \frac{\sqrt{\pi^{\mu\nu}\pi_{\mu\nu}}}{p}$$

Solution to Relaxation Time Equations:

- **Can't believe hydrodynamics too early:**

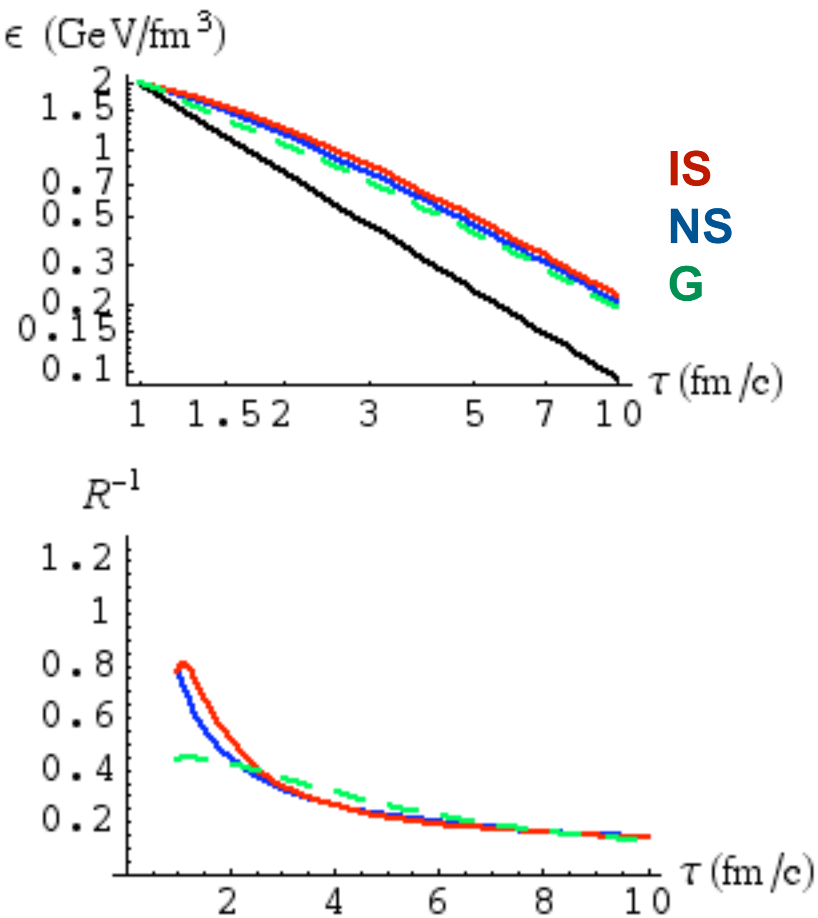
$\tau_0 = 1.5$ fm/c $\qquad\qquad$ $\tau_0 = 0.5$ fm/c



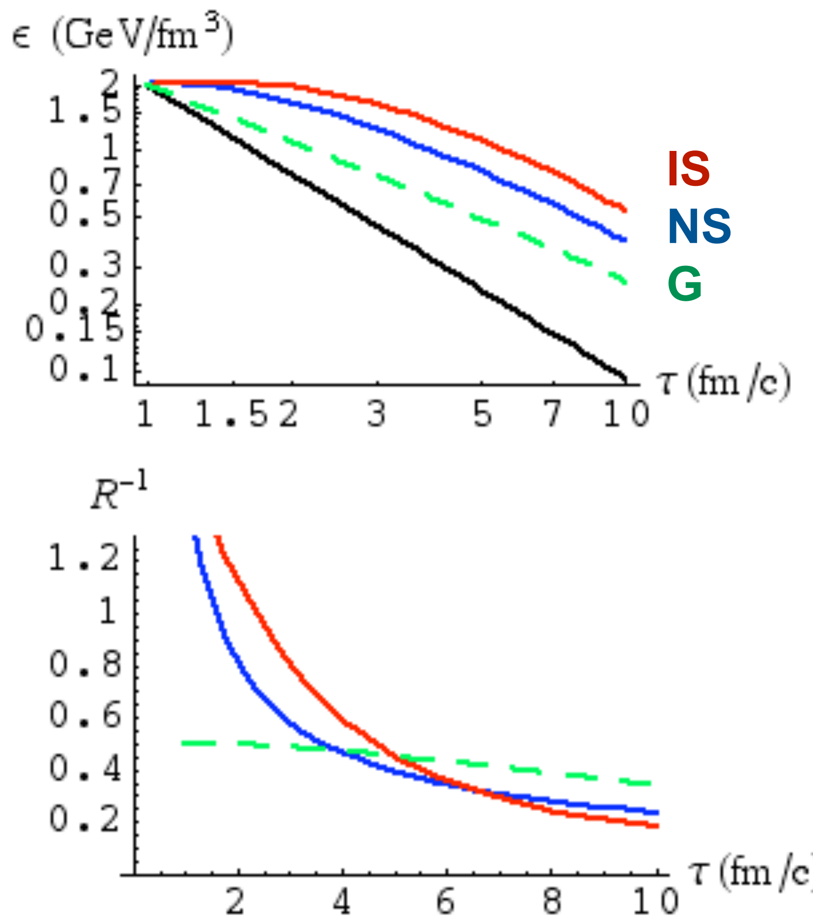
Solution to Relaxation Time Equations:

- **Can't believe hydrodynamics when viscosity becomes too large:**

$$\frac{\eta}{s} = 0.2$$



$$\frac{\eta}{s} = 0.5$$



Expect agreement between NS, IS, Generic:

- Hydro not started too early
- When viscosity is small enough
- Hydro not run too late
  - When transverse flow included gradients become large at larger radii
- When viscous terms become large models disagree with each other and NS

$$R^{-1} = \frac{\sqrt{\pi^{\mu\nu}\pi_{\mu\nu}}}{p} > 0.5 \qquad T^{ij} \neq \eta\left(\partial^i v^j + \partial^j v^i - \frac{2}{3}\delta^{ij}\partial_r v^r\right)$$

Freezeout is signaled by the equations

Viscous Corrections to Spectra: (M. Le Bellac, Arnold, Moore, Yaffe)

- Assume specific form for *f* close to equilibrium: $f \approx f_0 + \frac{\partial f_0}{\partial \varepsilon} \Phi \frac{\partial u_x}{\partial y}$

- Subst. into Boltzmann eqn gives eqn for η: $Df \equiv \frac{\partial f}{\partial t} + \vec{v} \cdot \frac{\partial f}{\partial \vec{x}} + \vec{F} \cdot \frac{\partial f}{\partial \vec{p}} = C[f]$

- Starting point point for variational approach: $\Phi = A(p)\, p_x p_y$

- A(p)=const agrees with exact results within 1%:

- Definition of pressure tensor: $P^{xy} = -\eta \frac{\partial u_x}{\partial y} = \int \frac{d^3 p}{(2\pi)^3} p_x v_y \frac{\partial f_0}{\partial \varepsilon} \Phi \frac{\partial u_x}{\partial y}$

- Solving for A from P_{xy} above gives: $A = \frac{\pi^4}{90 T^2 \zeta(5)} \frac{\eta}{s}$ and resulting form of *f* is:

$$\boxed{f \approx f_0 + \frac{3\Gamma}{8T^2} f_0 (1 + f_0)\, p^\mu p^\nu \langle \nabla_\mu u_\nu \rangle}$$

Viscous Corrections to Spectra:

- Viscosity changes the thermal distribution function:

$$f \longrightarrow f_0 + \delta f \qquad \delta f \sim \frac{p^i p^j}{T^2} \pi_{ij}$$

- The viscous corrections grow with momentum:

$$\delta f \sim \frac{l_{mfp}}{L} \left(\frac{p_T}{T} \right)^2$$

- Include viscous correction when computing freezeout integrals:

$$E \frac{dN}{d^3 p} = \int p^\mu d\Sigma_\mu \left(f_0 + \delta f \right)$$

1+1D Bjorken Expansion:

$$\partial_\tau T^{00} + \partial_r T^{01} = \frac{-1}{\tau}\left[T^{00} + \tilde{P}^{33}\right] - \frac{1}{r}\left[T^{01}\right]$$
$$\partial_\tau T^{01} + \partial_r T^{11} = \frac{-1}{\tau}\left[T^{01}\right] - \frac{1}{r}\left[T^{11} - \tilde{P}^{22}\right]$$

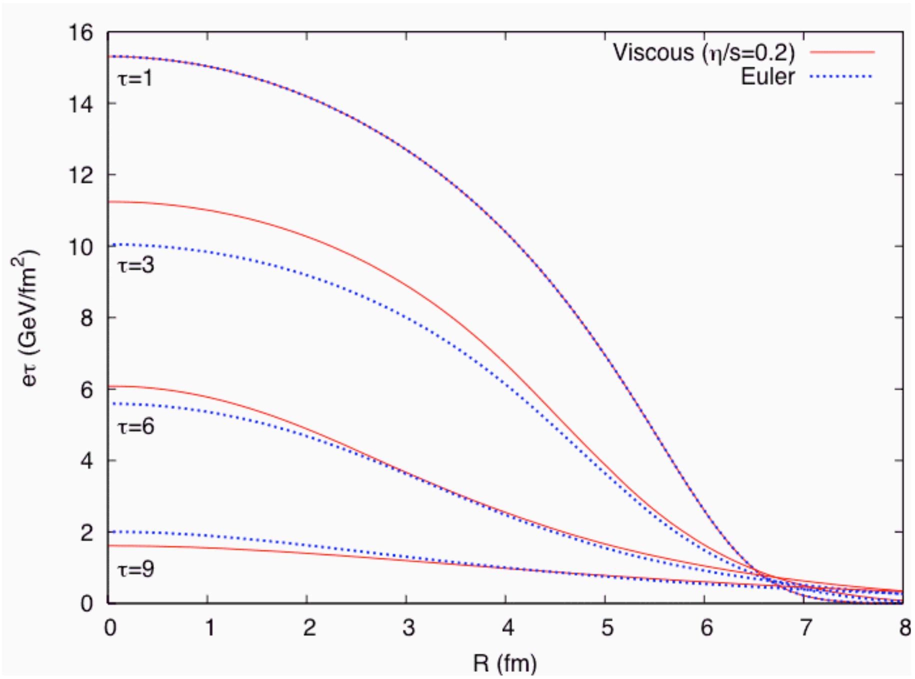
$$\partial_\tau c^{11} + v\partial_r c^{11} - \frac{2}{\gamma}\left[(1 - c^{11})\partial_r u^1 + c^{01}\partial_r u^0\right] = \frac{-1}{\gamma\tau_0}\bar{c}^{11} - \frac{1}{\gamma\tau_2}\mathring{c}^{11}$$
$$\partial_\tau \tilde{c}^{22} + v\partial_r \tilde{c}^{22} + \frac{2v}{r}(\tilde{c}^{22} - c^{11}) + \frac{2}{r}c^{10} = \frac{-1}{\gamma\tau_0}\bar{\tilde{c}}^{22} - \frac{1}{\gamma\tau_2}\mathring{\tilde{c}}^{22}$$
$$\partial_\tau \tilde{c}^{33} + v\partial_r \tilde{c}^{33} + \frac{2}{\tau}(\tilde{c}^{33} + c^{00}) - \frac{2v}{\tau}c^{10} = \frac{-1}{\gamma\tau_0}\bar{\tilde{c}}^{33} - \frac{1}{\gamma\tau_2}\mathring{\tilde{c}}^{33}$$

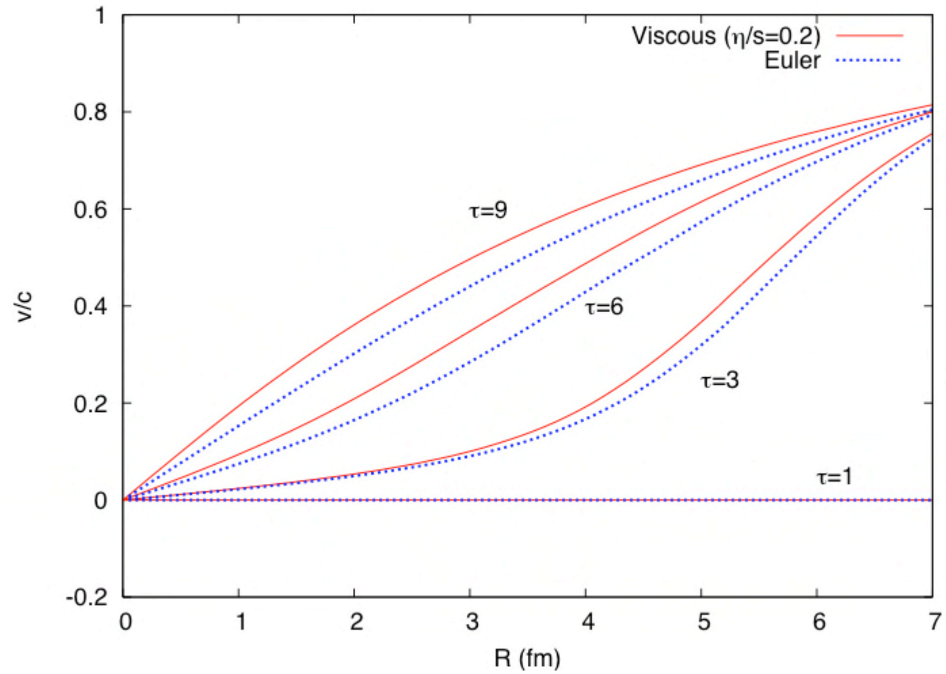
- Have algebraic system of equations for c
- 2nd Order RK scheme by L. Pareschi: can handle stiff and un-stiff source terms

- Steps:
  - IC: Mechanical force tensor takes equilibrium values at τ_0
  - Freezeout signaled by equations when viscous terms is about half the pressure
  - Compute Spectra

- Now some results….

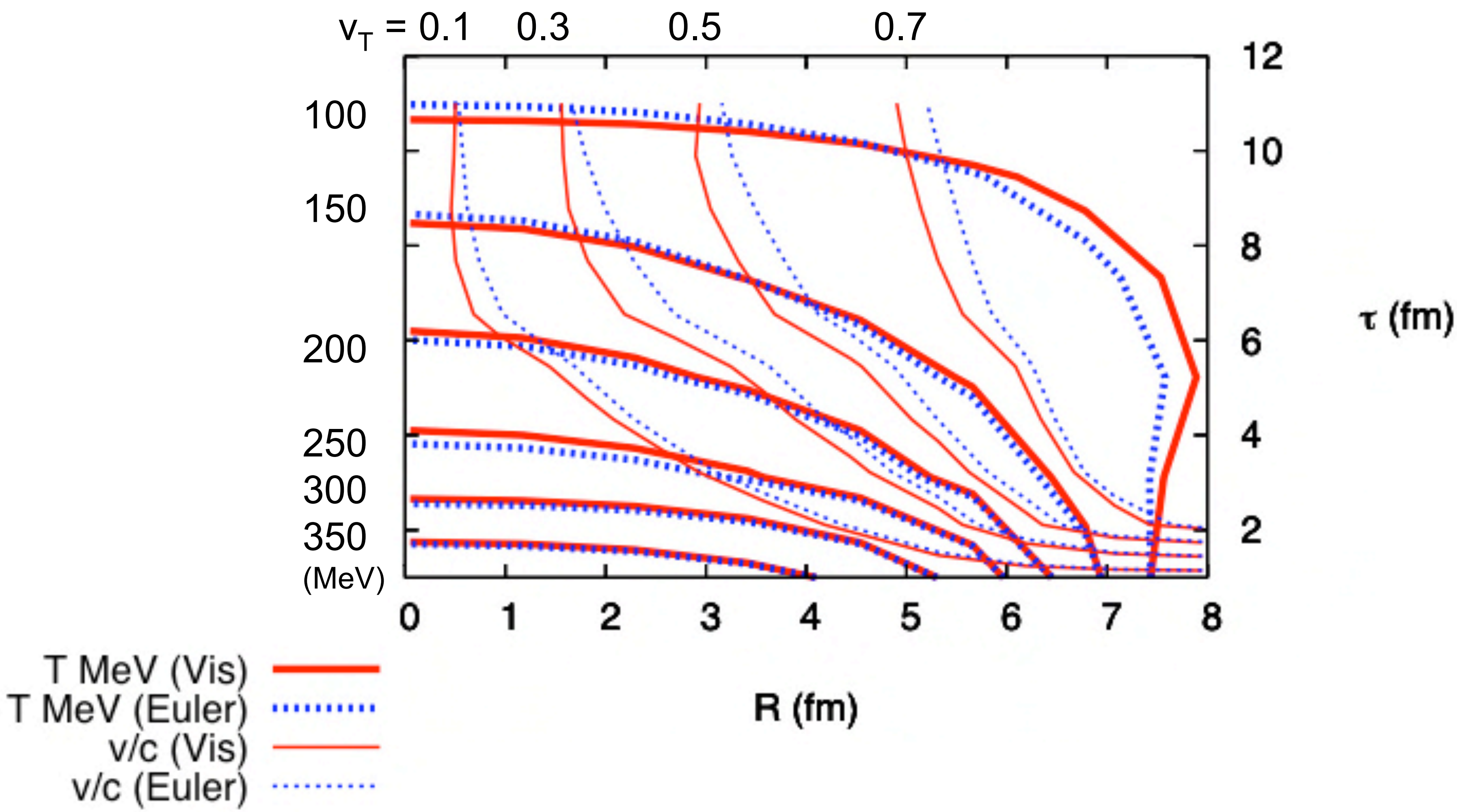
Result: 1+1D Bjorken





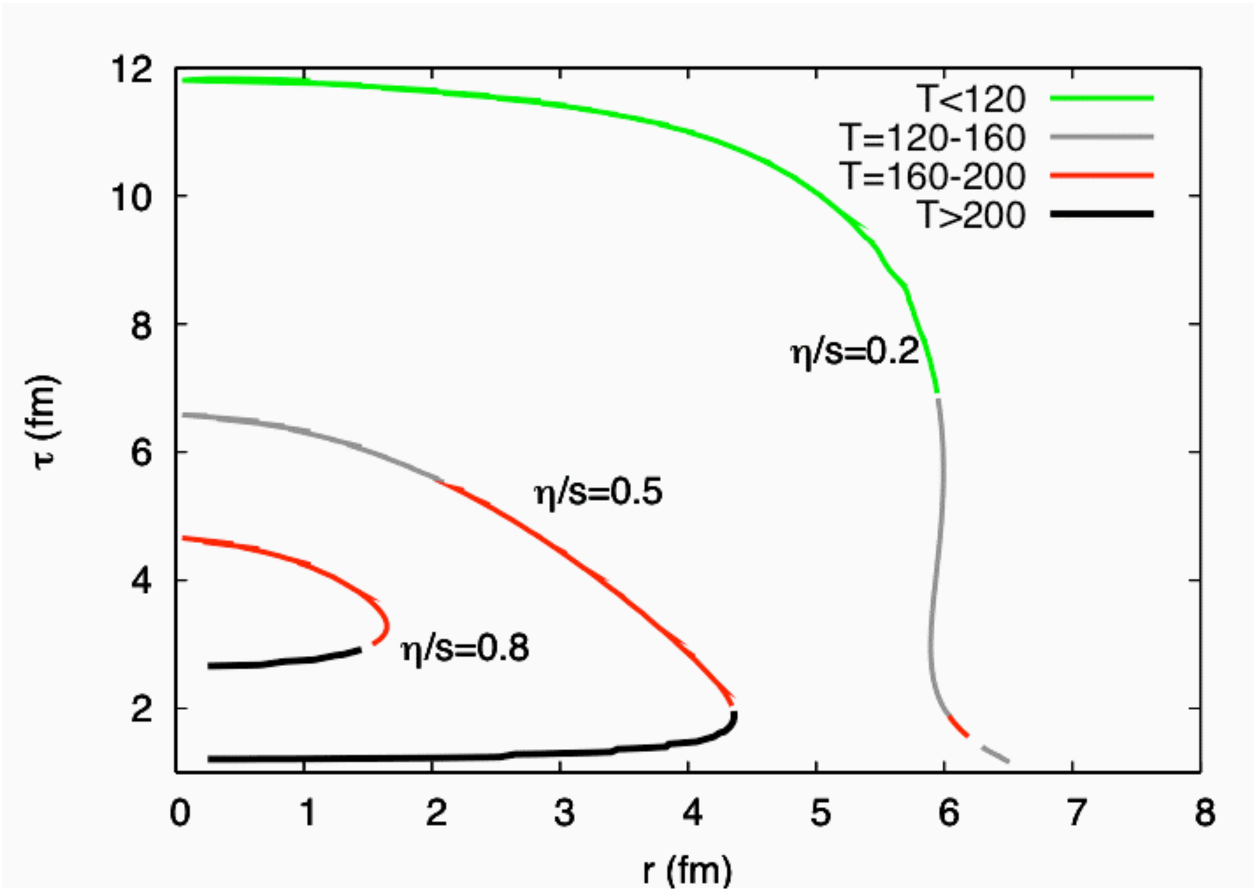
- Viscous solution does less longitudinal work
- The transverse pressure is larger leading to large transverse velocities
- These larger velocities result in a quicker reduction in energy density

Temperatures:



Viscous corrections to not change ideal solution much

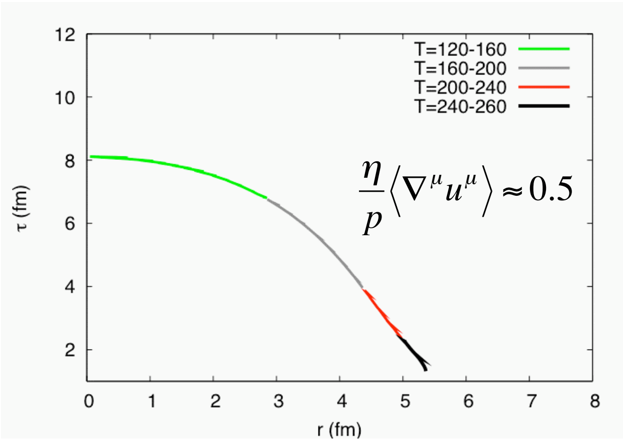
Freezeout: $\frac{\eta}{p}\left\langle \nabla^{\mu} u^{\mu} \right\rangle \approx 1$



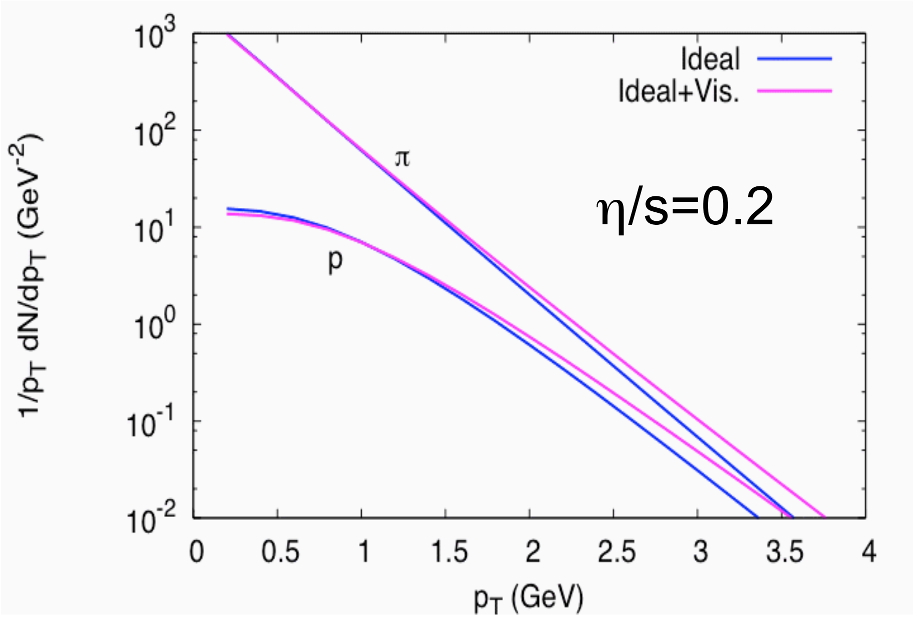
- Space-time volume where hydro is applicable set by η/s
- Freezeout surface is not an isotherm

Particle Spectra:

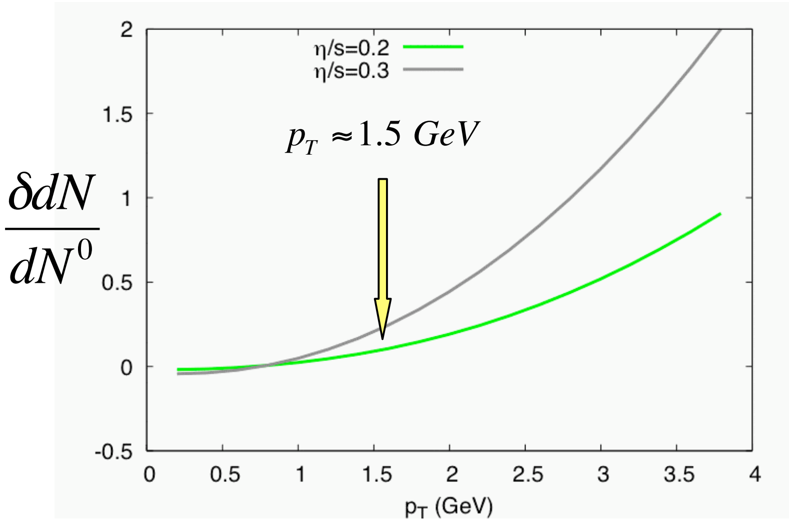
- 1. **Modification to freezeout surface:**



- 2. **Viscous correction to distribution function:** $f \longrightarrow f_0 + p^i p^j \langle \partial_i v_j \rangle$



% Correction



1+2D Bjorken Expansion:

$$\left.\begin{aligned} \partial_\tau T^{00} + \partial_x T^{01} + \partial_y T^{02} &= \frac{-1}{\tau}\left[T^{00} + \tau^2 P^{33}\right] \\ \partial_\tau T^{10} + \partial_x T^{11} + \partial_y T^{12} &= \frac{-1}{\tau} T^{10} \\ \partial_\tau T^{20} + \partial_x T^{21} + \partial_y T^{22} &= \frac{-1}{\tau} T^{20} \end{aligned}\right\} \text{Ideal Hydro Equations}$$

Relaxation Equations:

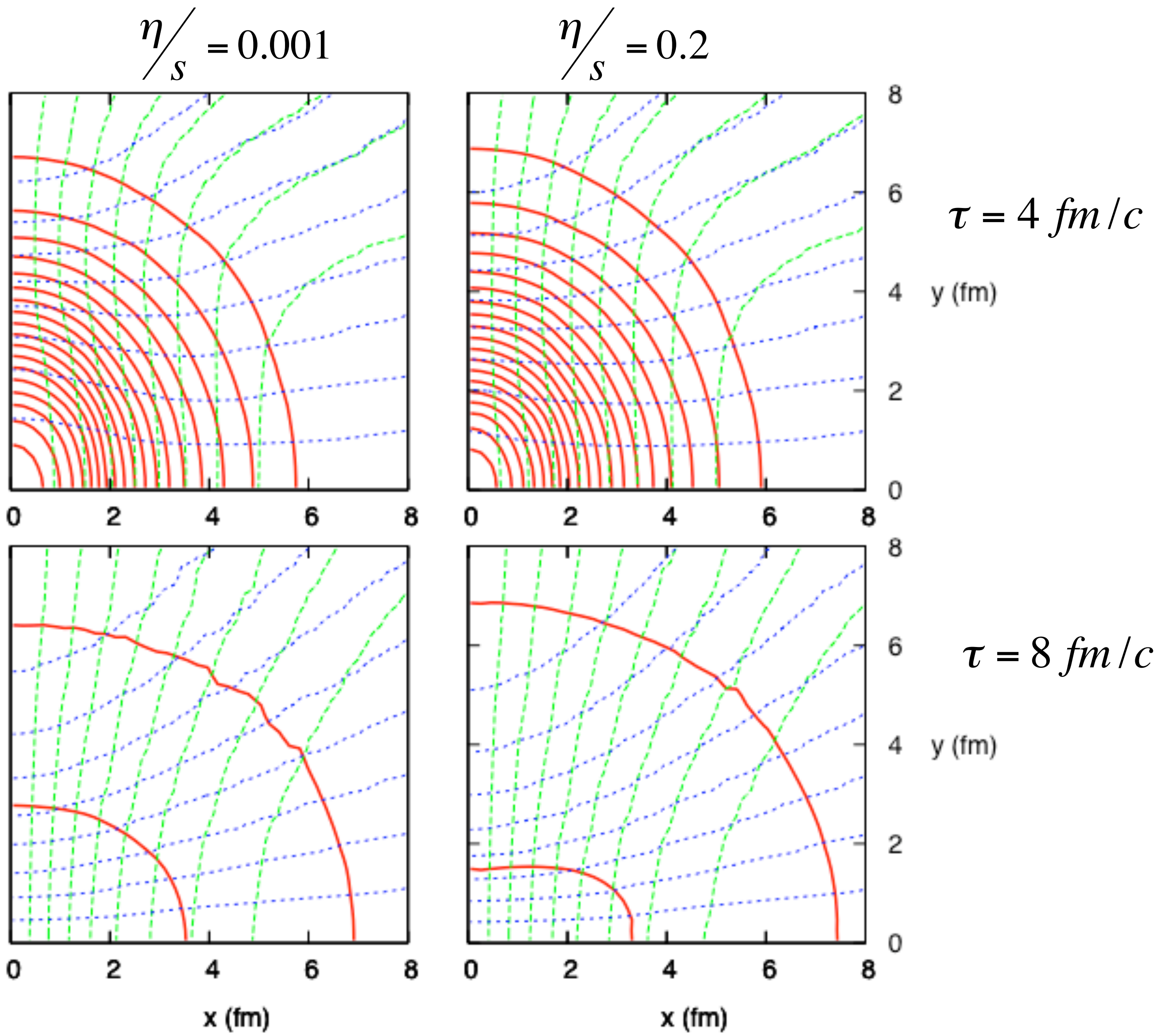
$$(\partial_\tau + v_x \partial_x + v_y \partial_y) c^{11} + \frac{2}{\gamma}\left[(c^{11} - 1)\partial_x v_x + c^{12}\partial_x v_y\right] = \frac{-1}{\gamma \tau_0}\bar{c}^{11} - \frac{1}{\gamma \tau_2}\mathring{c}^{11} \tag{46}$$

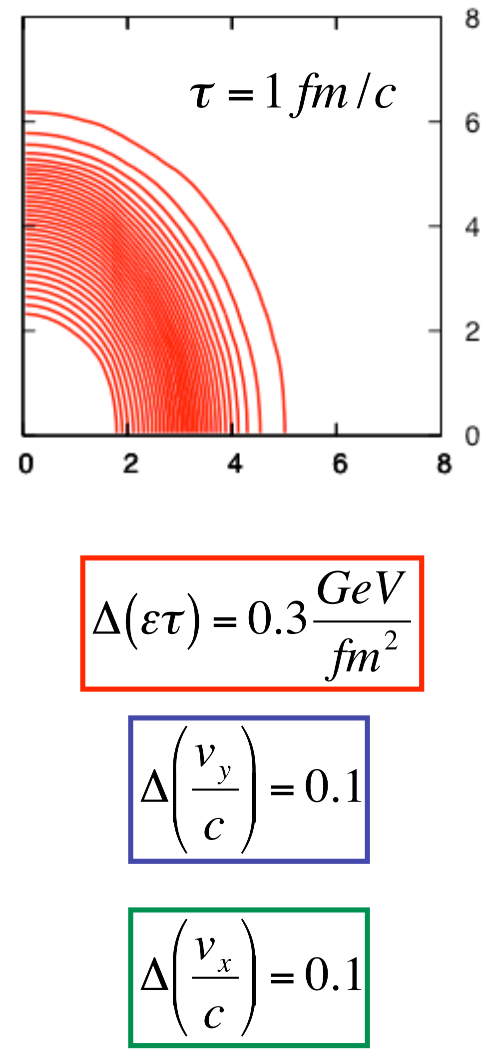
$$(\partial_\tau + v_x \partial_x + v_y \partial_y) c^{22} + \frac{2}{\gamma}\left[(c^{22} - 1)\partial_y v_y + c^{21}\partial_y v_x\right] = \frac{-1}{\gamma \tau_0}\bar{c}^{22} - \frac{1}{\gamma \tau_2}\mathring{c}^{22} \tag{47}$$

$$(\partial_\tau + v_x \partial_x + v_y \partial_y) \tilde{c}^{33} + \frac{2}{\tau}(\tilde{c}^{33} + c^{00} - v_x c^{10} - v_y c^{02}) = \frac{-1}{\gamma \tau_0}\bar{\tilde{c}}^{33} - \frac{1}{\gamma \tau_2}\mathring{\tilde{c}}^{33} \tag{48}$$

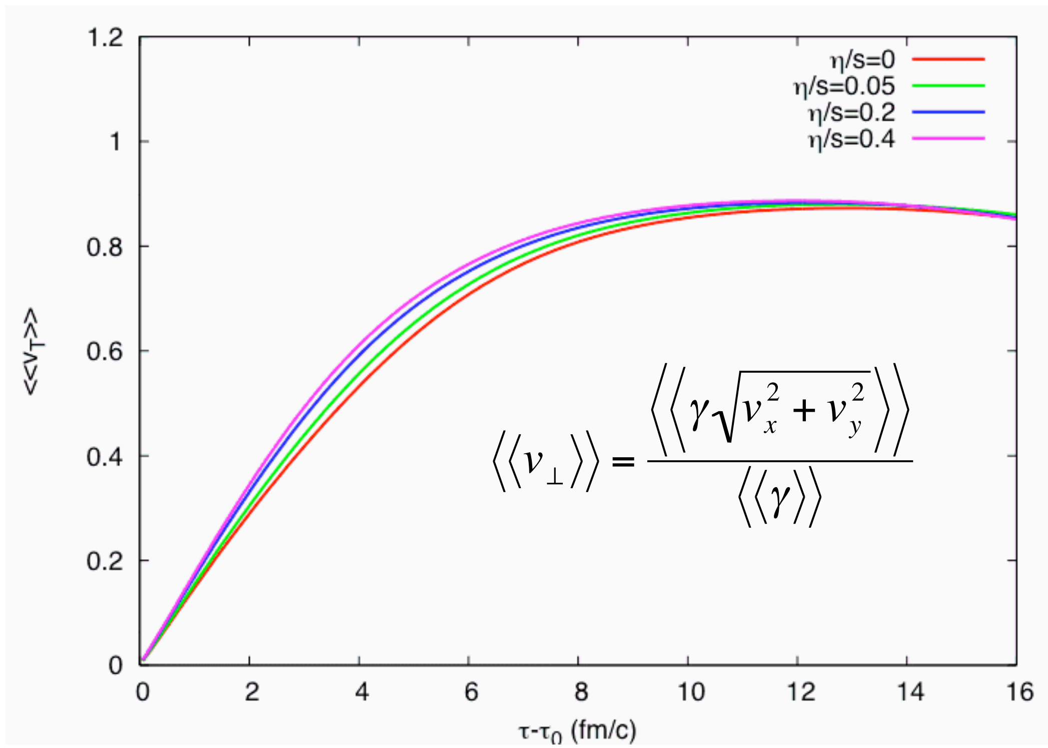
$$(\partial_\tau + v_x \partial_x + v_y \partial_y) c^{12} + c^{12}(\partial_x v_x + \partial_y v_y) + (c^{22} - 1)\partial_x v_y + (c^{11} - 1)\partial_y v_x = \frac{-1}{\gamma \tau_0}\bar{c}^{12} - \frac{1}{\gamma \tau_2}\mathring{c}^{12} \tag{49}$$

1+2D Bjorken Expansion: Au-Au (b=6 fm)

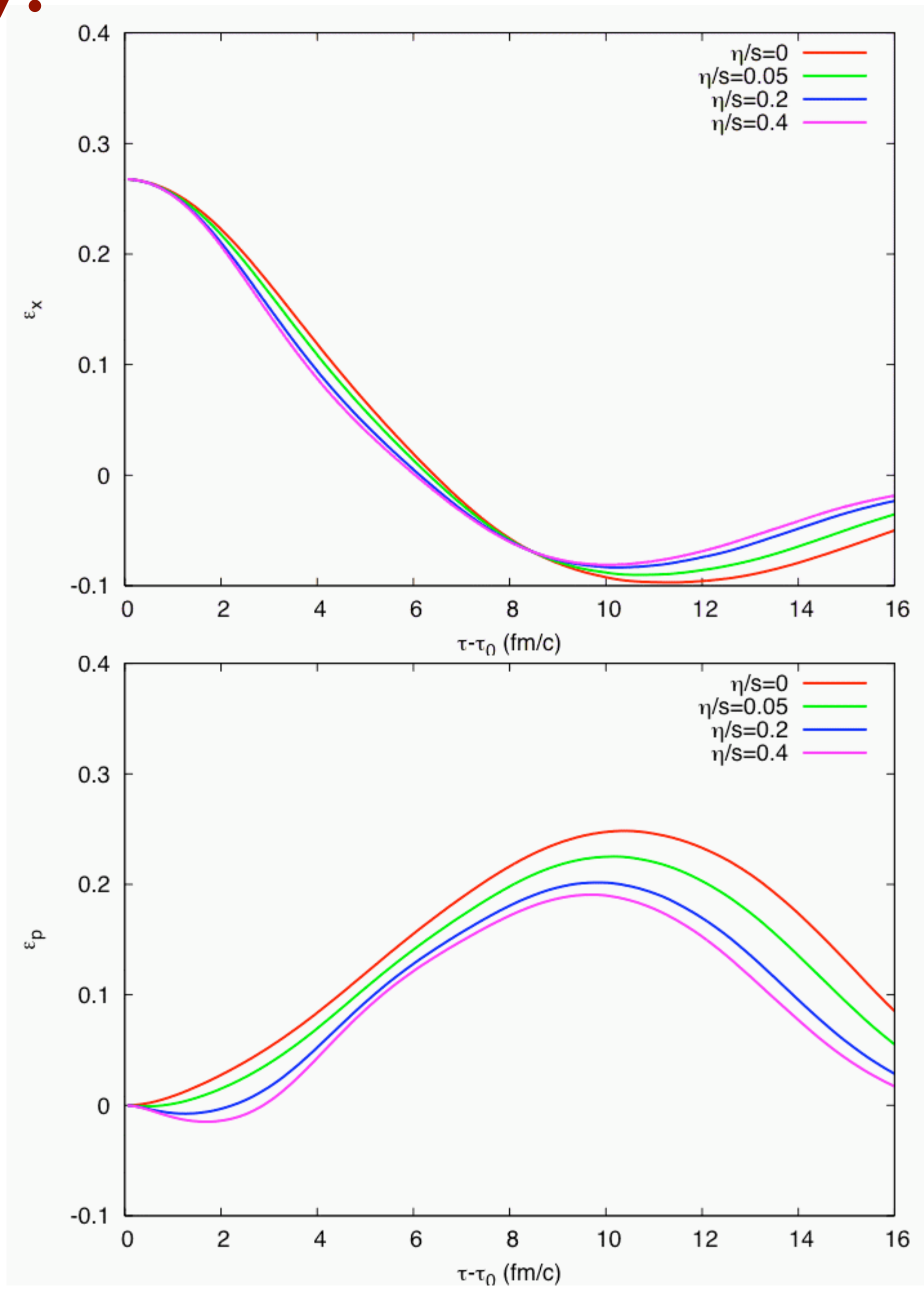




Transverse Flow:



Anisotropy:



$$\varepsilon_x = \frac{\langle\langle y^2 - x^2 \rangle\rangle}{\langle\langle y^2 + x^2 \rangle\rangle}$$

$$\varepsilon_p = \frac{\langle\langle T^{xx} - T^{yy} \rangle\rangle}{\langle\langle T^{xx} + T^{yy} \rangle\rangle}$$

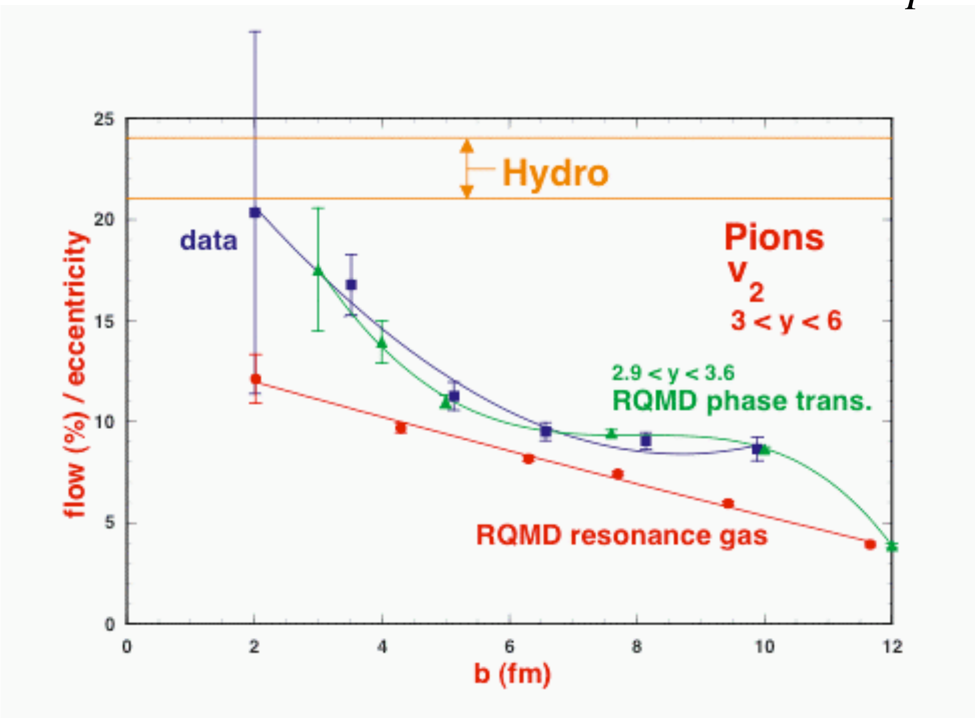
"Elliptic Flow"

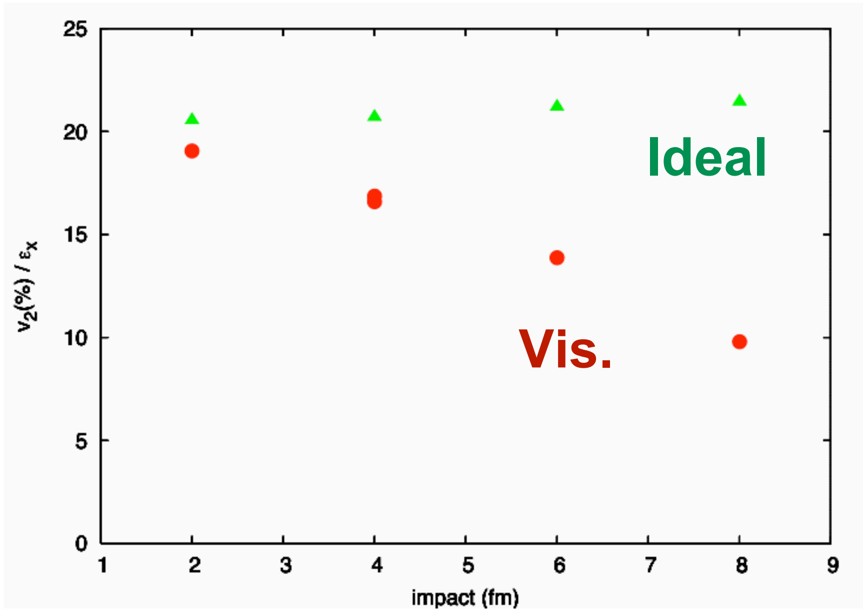
A.M. Poskanzer, et al., Nucl. Phys. A661, 341c (1999).
P.F. Kolb, J. Sollfrank, P.V. Ruuskanen and U. Heinz, Nucl. Phys. A661, 349c (1999).

$$v_2 \approx \frac{1}{2}\varepsilon_p(\tau_f)$$

$$\text{Ideal:}\quad \tau_f = \tau_f(T = 155\ \text{MeV})$$

$$\text{Vis:}\quad \tau_f = \tau_f\left(\frac{\eta}{p}\left\langle \nabla^\mu u^\nu \right\rangle \approx 0.5\right)$$





Conclusions / Summary:

- Progress being made with Viscous Hydro
  - Much more to be done

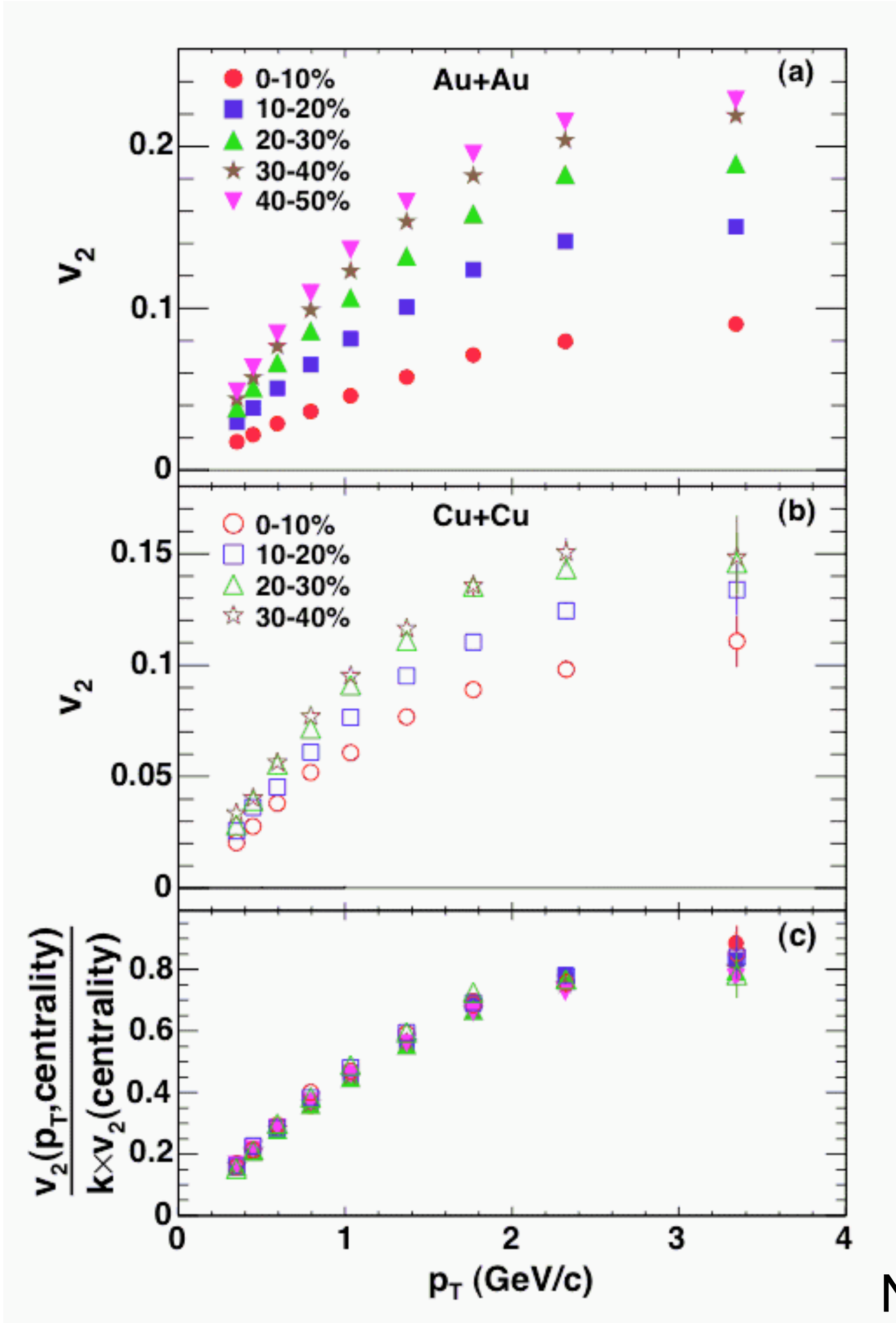
- Viscous effects in HIC
  - Ideal hydro solution not modified significantly
  - Signals the boundary of applicability
    - Tells us where to believe ideal hydro
  - Hope to constrain viscosity from v_2

- Viscous effects in general
  - Needs careful implementation
    - Viscous corrections may become large

- HIC great testing ground for understanding viscous hydro

✦ Backup Slides

Copper Data:



Nucl-ex/0608033

Evolution

- 1D Bjorken Expansion

$$T \propto \frac{1}{\tau^{1/3}} \quad \left(\frac{1}{\tau^{1/4}}\right) \qquad \tau n \propto \tau^0 \quad \left(\tau^{1/4}\right)$$

- Scale Invariant Cross section: $\sigma \sim \frac{\alpha_S^2}{T^2}$

$$\frac{\Gamma_s}{\tau} \sim \frac{1}{\tau n \sigma} \sim \frac{T^2}{\tau n} \sim \frac{1}{\tau^{2/3}} \quad \left(\frac{1}{\tau^{3/4}}\right)$$

Rapid Thermalization

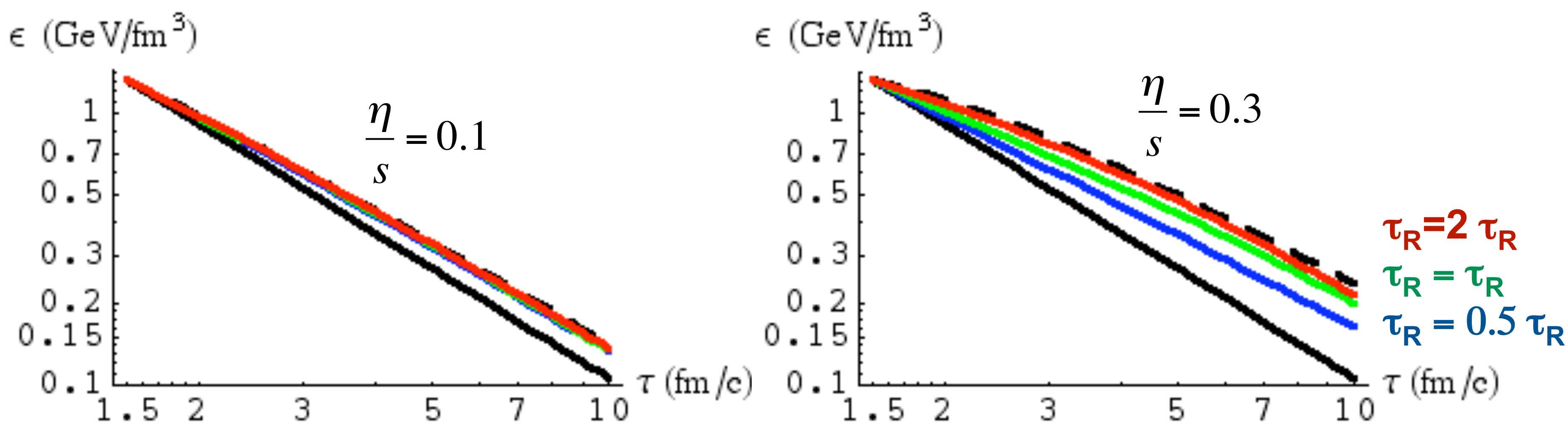
- Constant Cross section: $\sigma \sim const.$

$$\frac{\Gamma_s}{\tau} \sim \frac{1}{\tau n \sigma} \sim \frac{1}{\tau n} \sim 1 \quad \left(\frac{1}{\tau^{1/4}}\right)$$

Const (Slow) Thermalization

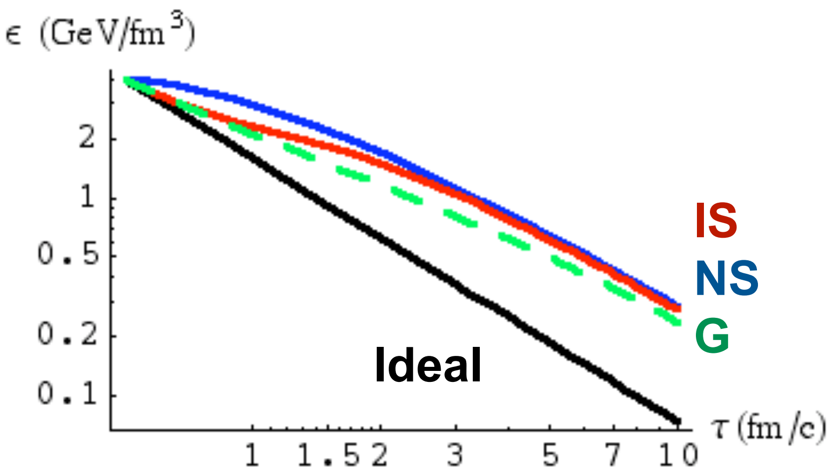
Solution to Relaxation Time Equations:

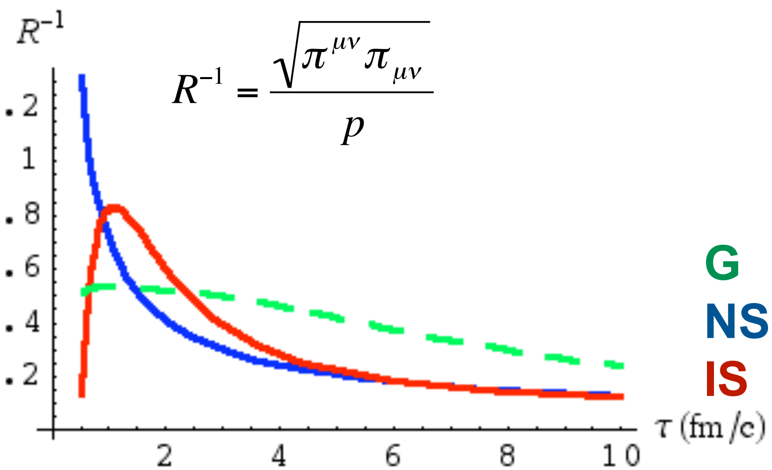
- **Relaxation time does not change solution much if viscous corrections are small:**



Another Look at Bjorken Expansion

$$\frac{\eta}{s} = 0.2 \qquad \tau_R = 2\eta \frac{3}{4p} \qquad \pi_0^{2nd} = \frac{1}{10}\pi_0^{1st}$$

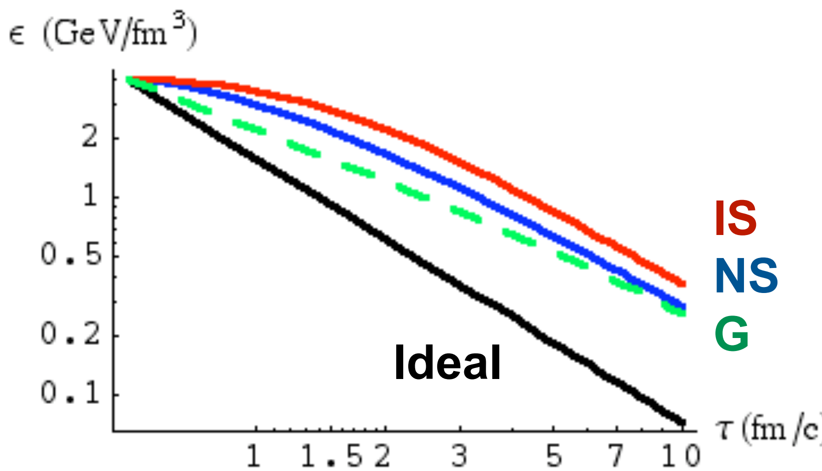


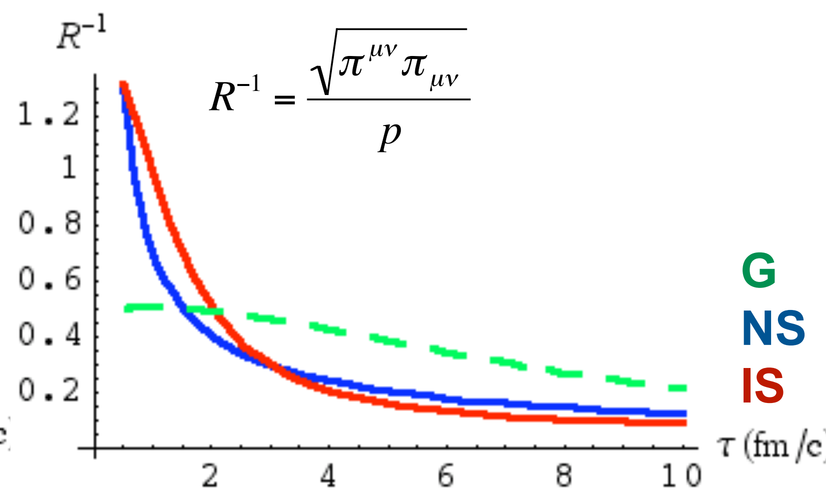


- Different IC did not effect Generic Solution
- IS now gives reasonable result but this is not the IC we really want.
- Again R becomes large at large radii when transverse flow present and varying IC won’t fix this.

Another Look at Bjorken Expansion

$$\frac{\eta}{s} = 0.2 \qquad \tau_R = 2\eta\frac{3}{4p} \qquad \pi_0^{2nd} = \pi_0^{1st}$$





- Viscous corrections are too large in NS and IS
- Limiting c in Generic keeps viscous corrections manageable
- When including transverse flow Vis. Corrections also grow at large R