Quantum Information

A brief introduction towards quantum teleportation

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Overview

- Qubit
- Measurement
- Schrodinger's Cat
- Two Qubits
- Quantum Logic Gates
  - NOT, Z, H – One qubit
  - CNOT – Two qubits
  - Combining Logic Gates → Bell States
- Quantum Teleportation

Qubits

- Smallest unit of quantum information
- Analogous to the bit in classical information theory: 0, 1
- Represented as a vector in a two dimensional Hilbert space: $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$

  (superposition)
- Magnitude squared of the complex coefficients must sum to 1 (probability rule)

Measurement

- Measurement collapses the possible states of the system

$$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle \xrightarrow{|\alpha|^2} |\psi\rangle = |0\rangle$$

$$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle \xrightarrow{|\beta|^2} |\psi\rangle = |1\rangle$$

Measurement

Schrodinger’s cat

Measurement

Decays → Geiger counter is triggered → acid is released → cat dies

Otherwise, cat lives.

Measurement

Schrodinger's cat

$$|cat\rangle = \alpha|dead\rangle + \beta|alive\rangle$$

$$|\alpha|^2 + |\beta|^2 = 1$$

Probability of being dead

Probability of being alive

Measurement

- Measurement collapses the possible states of the system

$$|\psi\rangle = \alpha|dead\rangle + \beta|alive\rangle \xrightarrow{|\alpha|^2} |\psi\rangle = |dead\rangle$$

$$|\psi\rangle = \alpha|dead\rangle + \beta|alive\rangle \xrightarrow{|\beta|^2} |\psi\rangle = |alive\rangle$$

Two Qubits

- In classical case, we can have:
  00, 01, 10, 11 – four configurations of two bits
- Similarly, in quantum case, we have:

$$|\psi\rangle = \alpha_{00}|00\rangle + \alpha_{01}|01\rangle + \alpha_{10}|10\rangle + \alpha_{11}|11\rangle$$

$$\sum_{x\in\{0,1\}^2} |\alpha_x|^2 = 1$$

Measurement

- Measurement collapses the possible states of the system
  $$|\psi\rangle = \alpha_{00}|00\rangle + \alpha_{01}|01\rangle + \alpha_{10}|10\rangle + \alpha_{11}|11\rangle$$
- For the first qubit (in the two qubit system), there is a $|\alpha_{00}|^2 + |\alpha_{01}|^2$ chance of getting a 0
- After measurement (and say we get 0), system collapses to:
  $$|\psi\rangle = \frac{1}{\sqrt{|\alpha_{00}|^2 + |\alpha_{01}|^2}}\left(\alpha_{00}|00\rangle + \alpha_{01}|01\rangle\right)$$

Quantum Logic Gates

- Classical NOT gate: 0 → 1, 1 → 0
- Quantum NOT gate:

$$|\psi_0\rangle = \alpha|0\rangle + \beta|1\rangle \longrightarrow |\psi_1\rangle = \beta|0\rangle + \alpha|1\rangle$$

$$\begin{bmatrix} \alpha \\ \beta \end{bmatrix} \longrightarrow \begin{bmatrix} \beta \\ \alpha \end{bmatrix}$$

Quantum Logic Gates

- Mathematically, how does this happen?
- Matrix multiplication!

$$X = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

$$X \begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \begin{bmatrix} \beta \\ \alpha \end{bmatrix}$$

Operators

- X is an operator
- Only constraint is unitarity: $XX^t = I$

$$X = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}, \quad X^t = \begin{bmatrix} \overline{a_{11}} & \overline{a_{21}} & \cdots & \overline{a_{n1}} \\ \overline{a_{12}} & \overline{a_{22}} & \cdots & \overline{a_{n2}} \\ \vdots & \vdots & \ddots & \vdots \\ \overline{a_{1n}} & \overline{a_{2n}} & \cdots & \overline{a_{nn}} \end{bmatrix}$$

Operators

- X is an operator
- Only constraint is unitarity: $XX^t = I$
- Invertibility of unitary matrices are representative of the reversibility of operations in quantum information
- Any unitary matrix can specify a quantum logic gate

More Logic Gates

- Z gate: $$Z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

Leaves $|0\rangle$ unchanged, changes the sign of $|1\rangle$

$$Z \begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \end{bmatrix} \longrightarrow \begin{bmatrix} \alpha \\ -\beta \end{bmatrix}$$

More Logic Gates

- Hadamard gate: $H = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$

Turns $|0\rangle$ to $\frac{|0\rangle + |1\rangle}{\sqrt{2}}$, and $|1\rangle$ to $\frac{|0\rangle - |1\rangle}{\sqrt{2}}$

$$H\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$$

More Logic Gates

- Hadamard gate: $H = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$

Turns $|0\rangle$ to $\frac{|0\rangle + |1\rangle}{\sqrt{2}}$, and $|1\rangle$ to $\frac{|0\rangle - |1\rangle}{\sqrt{2}}$

$$H\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{bmatrix}$$

More Logic Gates

- In summary:

$$\alpha\,|0\rangle + \beta\,|1\rangle \quad \text{———}\boxed{X}\text{———} \quad \beta\,|0\rangle + \alpha\,|1\rangle$$

$$\alpha\,|0\rangle + \beta\,|1\rangle \quad \text{———}\boxed{Z}\text{———} \quad \alpha\,|0\rangle - \beta\,|1\rangle$$

$$\alpha\,|0\rangle + \beta\,|1\rangle \quad \text{———}\boxed{H}\text{———} \quad \alpha\,\frac{|0\rangle+|1\rangle}{\sqrt{2}} + \beta\,\frac{|0\rangle-|1\rangle}{\sqrt{2}}$$

Multiple Qubit Logic Gate

- Controlled-NOT: CNOT
- Has two input qubits: control qubit and target qubit
- If the control qubit is 0, target qubit is unaffected
- If the control qubit is 1, target qubit is flipped

$$|00\rangle \rightarrow |00\rangle; \ \ |01\rangle \rightarrow |01\rangle; \ \ |10\rangle \rightarrow |11\rangle; \ \ |11\rangle \rightarrow |10\rangle$$

CNOT

$$|00\rangle \rightarrow |00\rangle;\ \ |01\rangle \rightarrow |01\rangle;\ \ |10\rangle \rightarrow |11\rangle;\ \ |11\rangle \rightarrow |10\rangle$$

$$\begin{bmatrix} |00\rangle \\ |01\rangle \\ |10\rangle \\ |11\rangle \end{bmatrix}$$

$$\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} = |00\rangle \qquad \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} = |01\rangle \qquad \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} = |10\rangle \qquad \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} = |11\rangle$$

CNOT

$$|00\rangle \rightarrow |00\rangle;\ \ |01\rangle \rightarrow |01\rangle;\ \ |10\rangle \rightarrow |11\rangle;\ \ |11\rangle \rightarrow |10\rangle$$

$$\begin{bmatrix}1\\0\\0\\0\end{bmatrix}=|00\rangle \quad \begin{bmatrix}0\\1\\0\\0\end{bmatrix}=|01\rangle \quad \begin{bmatrix}0\\0\\1\\0\end{bmatrix}=|10\rangle \quad \begin{bmatrix}0\\0\\0\\1\end{bmatrix}=|11\rangle$$

$$\Rightarrow CNOT=\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&0&1\\0&0&1&0\end{bmatrix} \Rightarrow$$

$$\begin{bmatrix}1\\0\\0\\0\end{bmatrix}=|00\rangle \quad \begin{bmatrix}0\\1\\0\\0\end{bmatrix}=|01\rangle \quad \begin{bmatrix}0\\0\\0\\1\end{bmatrix}=|11\rangle \quad \begin{bmatrix}0\\0\\1\\0\end{bmatrix}=|10\rangle$$

Combining Logic Gates - Circuits

- Let's put a Hadamard gate followed by a CNOT gate
- Hadamard (on the first qubit) takes:

$$|00\rangle \rightarrow \left(\frac{|0\rangle + |1\rangle}{\sqrt{2}}\right)|0\rangle = \frac{|0\rangle|0\rangle + |1\rangle|0\rangle}{\sqrt{2}}$$

- Then CNOT gives the output:

$$\frac{|0\rangle|0\rangle + |1\rangle|0\rangle}{\sqrt{2}} = \frac{|00\rangle + |10\rangle}{\sqrt{2}} \rightarrow \frac{|00\rangle + |11\rangle}{\sqrt{2}}$$

Combining Logic Gates - Circuits

- Repeating for all the others, we get:

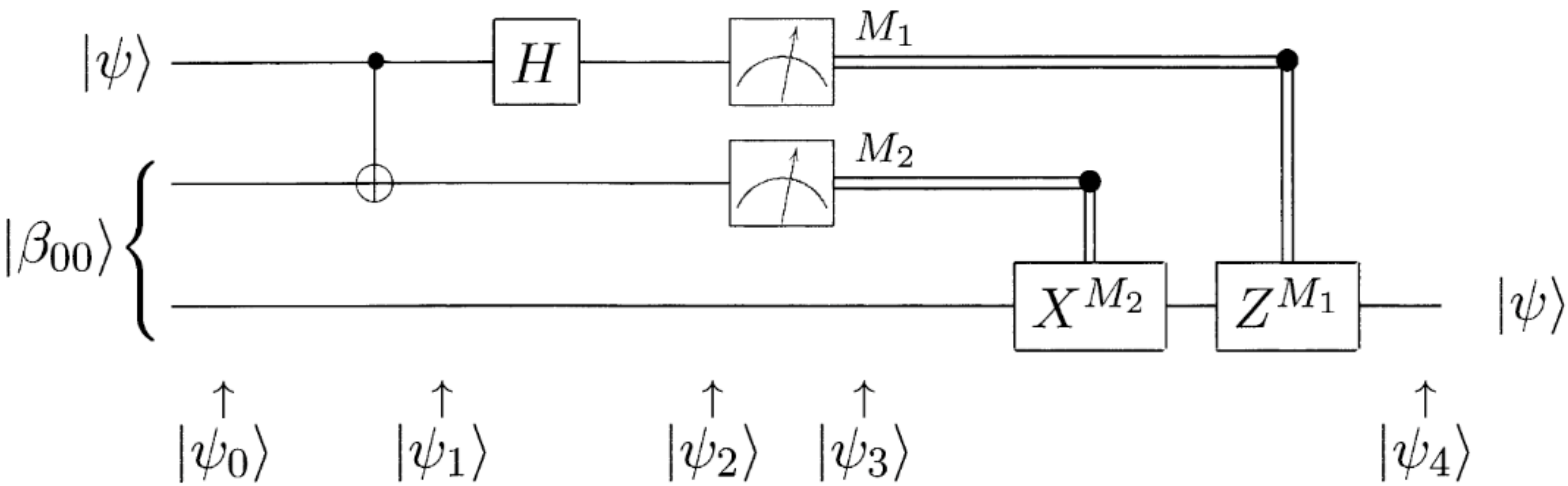
$$\text{Bell States}\begin{cases} |\beta_{00}\rangle = \dfrac{|00\rangle + |11\rangle}{\sqrt{2}}; \\ |\beta_{01}\rangle = \dfrac{|01\rangle + |10\rangle}{\sqrt{2}}; \\ |\beta_{10}\rangle = \dfrac{|00\rangle - |11\rangle}{\sqrt{2}}; \text{ and} \\ |\beta_{11}\rangle = \dfrac{|01\rangle - |10\rangle}{\sqrt{2}}, \end{cases}$$

Bell States

- Also known as EPR pairs – named after Bell, Einstein, Podolsky, and Rosen

In	Out
$\|00\rangle$	$(\|00\rangle + \|11\rangle)/\sqrt{2} \equiv \|\beta_{00}\rangle$
$\|01\rangle$	$(\|01\rangle + \|10\rangle)/\sqrt{2} \equiv \|\beta_{01}\rangle$
$\|10\rangle$	$(\|00\rangle - \|11\rangle)/\sqrt{2} \equiv \|\beta_{10}\rangle$
$\|11\rangle$	$(\|01\rangle - \|10\rangle)/\sqrt{2} \equiv \|\beta_{11}\rangle$

Quantum Teleportation



Quantum Teleportation

GOAL: Transmit a qubit -

$$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$$

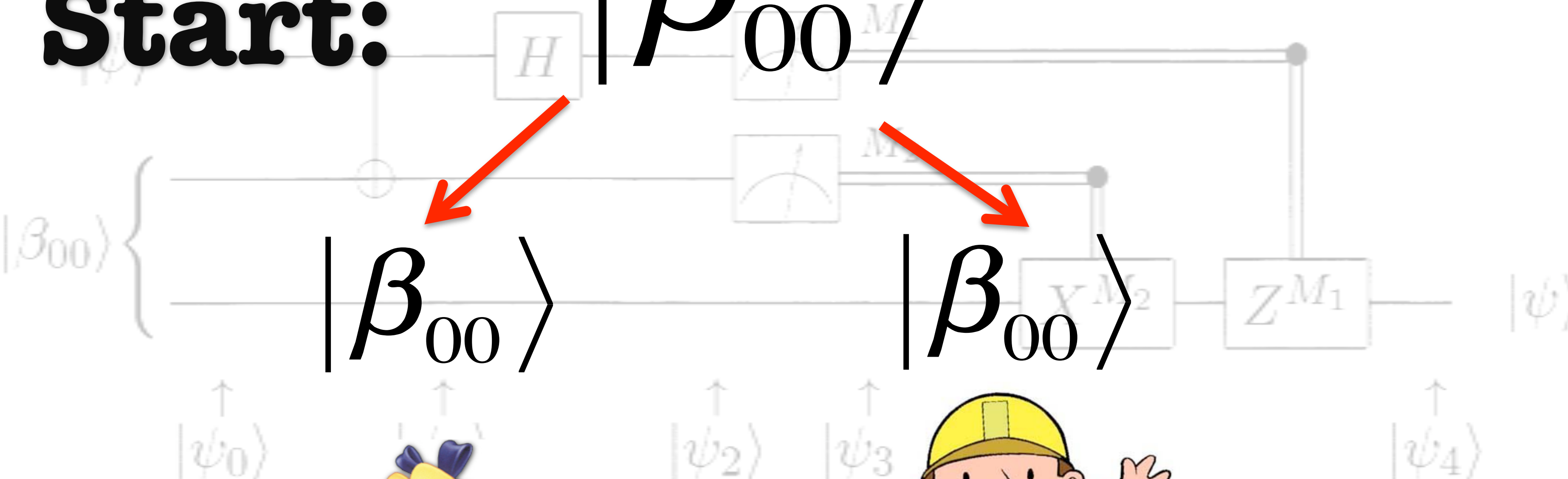
Alice

Bob

$|\psi\rangle$

Quantum Teleportation

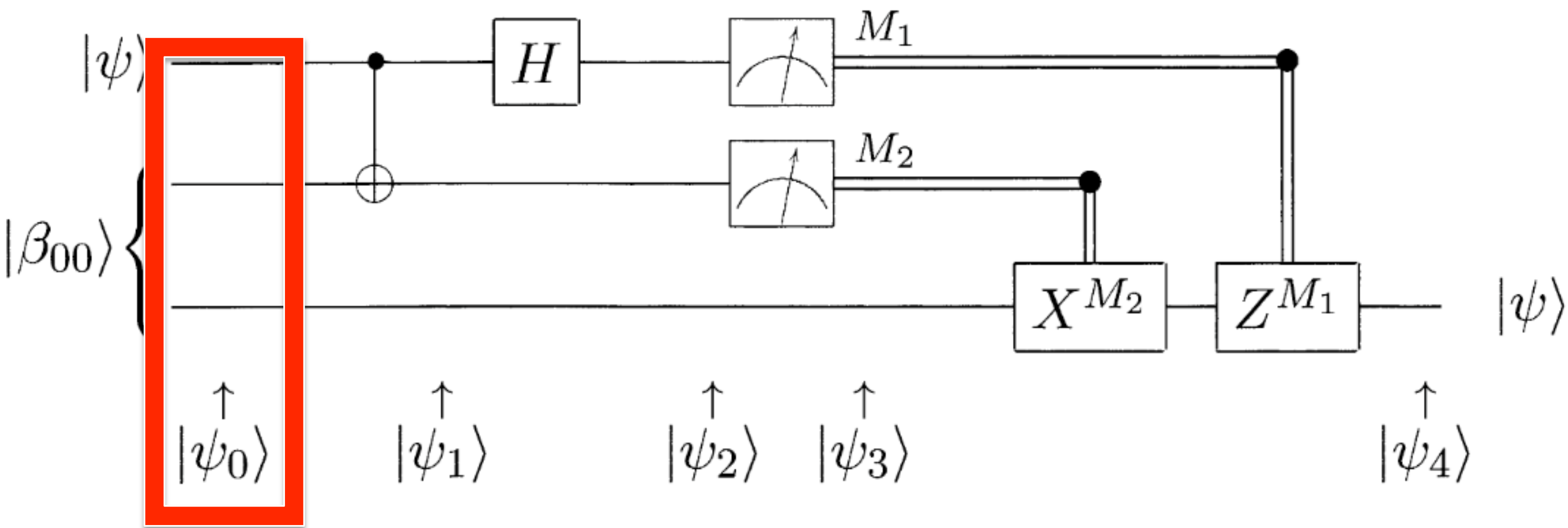
Start: $|\beta_{00}\rangle$



$|\beta_{00}\rangle$ $|\beta_{00}\rangle$

Quantum Teleportation

- Alice first interacts her qubit from the original EPR pair with the qubit that she wants to teleport:



Quantum Teleportation

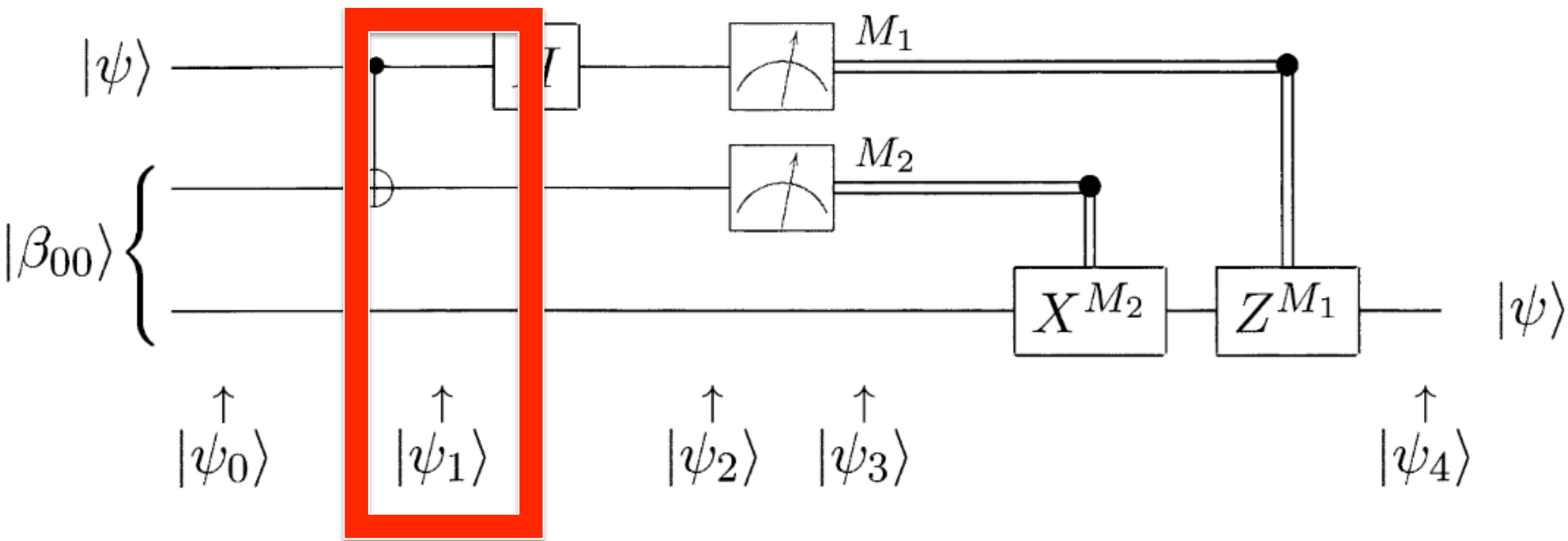
- Alice first interacts her qubit from the original EPR pair with the qubit that she wants to teleport:

$$|\psi_0\rangle = |\psi\rangle|\beta_{00}\rangle$$

$$= \frac{1}{\sqrt{2}}\left[\alpha|0\rangle(|00\rangle + |11\rangle) + \beta|1\rangle(|00\rangle + |11\rangle)\right]$$

$$\left|\psi_0\right\rangle = \frac{1}{\sqrt{2}}\left[\alpha|0\rangle(|00\rangle + |11\rangle) + \beta|1\rangle(|00\rangle + |11\rangle)\right]$$

- Alice then sends her qubits through a CNOT gate to get:



$$\left|\psi_0\right\rangle = \frac{1}{\sqrt{2}}\left[\alpha|0\rangle(|00\rangle + |11\rangle) + \beta|1\rangle(|00\rangle + |11\rangle)\right]$$

- Alice then sends her qubits through a CNOT gate to get:

$$\left|\psi_0\right\rangle \Rightarrow CNOT \Rightarrow \left|\psi_1\right\rangle$$

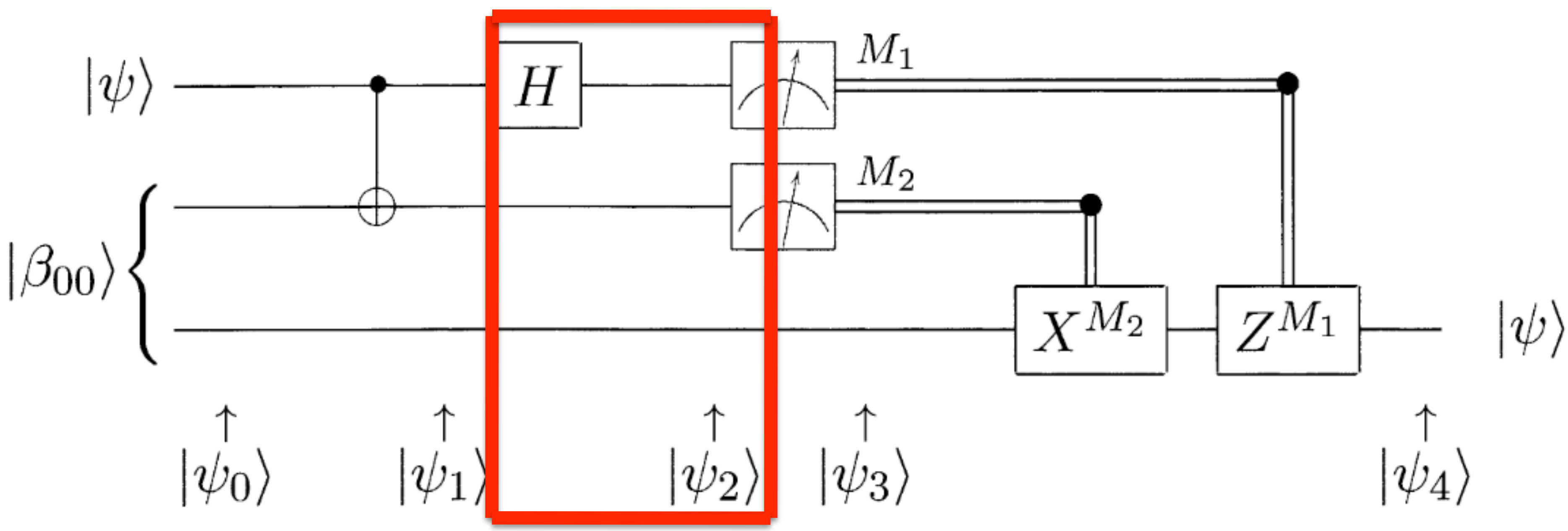
$$|\psi_1\rangle = \frac{1}{\sqrt{2}}\left[\alpha|0\rangle(|00\rangle + |11\rangle) + \beta|1\rangle(|10\rangle + |01\rangle)\right]$$

Keeps

Flips

$$|\psi_1\rangle = \frac{1}{\sqrt{2}}\left[\alpha|0\rangle(|00\rangle + |11\rangle) + \beta|1\rangle(|10\rangle + |01\rangle)\right]$$

- Finally, Alice sends her first qubit through a Hadamard gate to get:



$$|\psi_1\rangle = \frac{1}{\sqrt{2}}\left[\alpha|0\rangle(|00\rangle + |11\rangle) + \beta|1\rangle(|10\rangle + |01\rangle)\right]$$

- Finally, Alice sends her first qubit through a Hadamard gate to get:

$$|\psi_1\rangle \Rightarrow H - gate \Rightarrow |\psi_2\rangle$$

$$|\psi_2\rangle = \frac{1}{2}\left[\alpha(|0\rangle + |1\rangle)(|00\rangle + |11\rangle) + \beta(|0\rangle - |1\rangle)(|10\rangle + |01\rangle)\right]$$

Recall:

H-gate: Turns $|0\rangle$ to $\frac{|0\rangle + |1\rangle}{\sqrt{2}}$, and $|1\rangle$ to $\frac{|0\rangle - |1\rangle}{\sqrt{2}}$.

Quantum Teleportation

- Rearranging the terms of $|\psi_2\rangle$, we get:

$$|\psi_2\rangle = \frac{1}{2}\Big[|00\rangle\left(\alpha|0\rangle + \beta|1\rangle\right) + |01\rangle\left(\alpha|1\rangle + \beta|0\rangle\right) + |10\rangle\left(\alpha|0\rangle - \beta|1\rangle\right) + |11\rangle\left(\alpha|1\rangle - \beta|0\rangle\right)\Big]$$

Quantum Teleportation

- We now have four terms:

$$|\psi_2\rangle = \frac{1}{2}\Big[\boxed{|00\rangle}(\alpha|0\rangle + \beta|1\rangle) + \boxed{|01\rangle}(\alpha|1\rangle + \beta|0\rangle)$$
$$+ \boxed{|10\rangle}(\alpha|0\rangle - \beta|1\rangle) + \boxed{|11\rangle}(\alpha|1\rangle - \beta|0\rangle)\Big]$$

- Alice makes her measurement…

Quantum Teleportation

- Recall that measurement collapses the state of the qubit.
- Note that: $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$

$$|\psi_2\rangle = \frac{1}{2}\Big[\boxed{|00\rangle}\overbrace{(\alpha|0\rangle + \beta|1\rangle)} + |01\rangle(\alpha|1\rangle + \beta|0\rangle)$$
$$+ |10\rangle(\alpha|0\rangle - \beta|1\rangle) + |11\rangle(\alpha|1\rangle - \beta|0\rangle)\Big]$$

Quantum Teleportation

- If Alice measures 00, then Bob's qubit is in the state: $\alpha|0\rangle + \beta|1\rangle = |\psi\rangle$.
- By the same reasoning, if Alice measures 01, then Bob's qubit is in the state:

$$\alpha|1\rangle + \beta|0\rangle$$

$$|\psi_2\rangle = \frac{1}{2}\Big[|00\rangle\left(\alpha|0\rangle + \beta|1\rangle\right) + |01\rangle\left(\alpha|1\rangle + \beta|0\rangle\right) + |10\rangle\left(\alpha|0\rangle - \beta|1\rangle\right) + |11\rangle\left(\alpha|1\rangle - \beta|0\rangle\right)\Big]$$

Quantum Teleportation

$$00 \longmapsto |\psi_3(00)\rangle \equiv \Big[\alpha|0\rangle + \beta|1\rangle\Big]$$

$$01 \longmapsto |\psi_3(01)\rangle \equiv \Big[\alpha|1\rangle + \beta|0\rangle\Big]$$

$$10 \longmapsto |\psi_3(10)\rangle \equiv \Big[\alpha|0\rangle - \beta|1\rangle\Big]$$

$$11 \longmapsto |\psi_3(11)\rangle \equiv \Big[\alpha|1\rangle - \beta|0\rangle\Big]$$

Quantum Teleportation

- Bob must now correct his qubit state.
- If Alice measures 00, Bob does not have to do anything.
- Otherwise, Bob applies a combination of X and Z gates.
- For example, if Alice measures 01, then Bob's qubit is in the state:

$$\alpha|1\rangle + \beta|0\rangle$$

Quantum Teleportation

- To get back the original qubit, recall that the X gate (NOT gate) swaps the coefficient.
- So Bob can apply the X gate to:

$$\alpha|1\rangle + \beta|0\rangle$$

to get:

$$\alpha|0\rangle + \beta|1\rangle$$

The Original!

Quantum Teleportation

- How does changing Alice’s qubit affect Bob’s?
- Recall start (pg. 7) – they each get the same initial system from the EPR pair
- Referred to as entanglement
- There has already been a successful experiment in which 10^{12} atoms were transported
- Just for comparison, an average 70 kg person consists of approximately 7 x 10^{27}

Books To Read

- Quantum Computation and Quantum Information, by Nielsen and Chuang
- Foundations of Quantum Theory, by Preskill
- Information Theory and Quantum Physics, by Green
- Linear Algebra, by Hoffman and Kunze


WANTED
|DEAD⟩ OR |ALIVE⟩
OR
$\frac{1}{\sqrt{2}}(|DEAD\rangle + |ALIVE\rangle)$