Nuclear Liquid Gas Phase Transition and Nuclear Zipf Law

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Nuclear Zipf’s law has been proposed for the studies of the nuclear disassembly and liquid gas phase transition. In this report, firstly we use the isospin dependent lattice gas model and molecular dynamical model to define the nuclear Zipf-type plot according to the fragment mass (charge) hierarchy and predicts that there exists the nuclear Zipf law, namely the mean mass (charge) of the cluster shows inversely to its rank from the largest size to the smallest size, around the nuclear liquid gas phase transition. Secondly we show the experimental evidences for nuclear Zipf’s law in the quasi-projectiles fragmentation data of Ar beams by NIMROD multidetector as well as in the analysis of CERN emulsion data of Pb + Pb or Plastic at 158 AGeV or lower energies.

I. INTRODUCTION

Hot nuclei can be formed in energetic heavy ion collisions (HIC) and deexcite by different decay modes, such as evaporation and multifragmentation. Experimentally, this kind of multifragment emission was observed to evolve with beam energy (excitation energy, or nuclear temperature, ...). Multiplicity, N_{IMF}, of intermediate mass fragment (IMF) rises with beam energy, reaches a maximum, and finally falls to lower value. This phenomenon of the rise and fall of N_{IMF} may be related to the liquid gas phase transition in nuclear matter [1]. The onset of multifragmentation probably indicates the coexistence of liquid and gas phases. The mass (charge) distribution of IMF distribution can be expressed as power law with parameter τ. The minimum of τ, τ_{min}, occurs when the liquid gas phase transition takes place [2].

On the other hand, the caloric curve measurement can also provide useful information on the liquid gas phase transition [3–6]. The analysis of critical exponents provides an additional judgement of the critical behavior of finite nuclear system [7–12]. More observables are suggested to judge the liquid gas phase transition of nuclei [13–17].

In this report, the original Zipf’s law [18] has been introduced into the diagnosis of nuclear liquid gas phase transition and we called it as nuclear Zipf’s law. Zipf’s law has been known as a statistical phenomenon concerning the relation between English words and their frequency in literature in the field of linguistics [18]. The law states that, when we list the words in the order of decreasing population, the frequency of a word is inversely proportional to its rank [18]. This relation was found not only in linguistics but also in other fields of sciences. For instance, the law appeared in distributions of populations in cities, distributions of income of corporations, distributions of areas of lakes and cluster-size distribution in percolation process [19, 20]. The details for the proposal of nuclear Zipf’s law can be found in Ref. [21, 22]. In this report, we firstly define the nuclear Zipf plot for the fragment mass (charge) distribution and nuclear Zipf’s law in the simulation with help of the lattice gas model. Then we show the evidence of nuclear Zipf law around the liquid gas phase transition. Finally we give conclusion and outlook.

II. MODEL SIMULATIONS

The tools we will use here are the isospin dependent lattice gas model (LGM) and molecular dynamical model (MD). The lattice gas model was developed to describe the liquid-gas phase transition for atomic system by Lee and Yang [23]. The same model has already been applied to nuclear physics for isospin symmetrical systems in the grandcanonical ensemble [24] with a sampling of the canonical ensemble [25–31], and also for isospin asymmetrical nuclear matter in the mean field approximation [32]. In addition, a classical molecular dynamical model is used to compare its results with the results of lattice gas model. Here we will make a brief description for the models.

In the lattice gas model, A (= N + Z) nucleons with an occupation number s which is defined s = 1 (-1) for a proton (neutron) or s = 0 for a vacancy, are placed on the L sites of lattice. Nucleons in the nearest neighboring sites interact with an energy ϵ_{s,s’}. The Hamiltonian is written as E = \sum_{i=1}^{A} \frac{p^2_i}{2m} - \sum_{i<j} \epsilon_{s_i,s_j}s_is_j. The interaction constant ϵ_{s,s’} is chosen to be isospin dependent and be fixed to reproduce the binding energy of the nuclei [27]:

\begin{align}
\epsilon_{nn} &= \epsilon_{pp} = 0.\text{MeV}, \\
\epsilon_{pn} &= -5.33\text{MeV}.
\end{align}

(1)

A three-dimension cubic lattice with L sites is used. The freeze-out density of disassembling system is assumed to be \rho_f = \frac{2}{L^3}\rho_0, where \rho_0 is the normal nuclear density. The disassembly of the system is to be calculated at \rho_f, beyond which nucleons are too far apart to interact. Nucleons are put into lattice by Monte Carlo Metropolis sampling. Once the nucleons have been placed we also ascribe to each of them a momentum by Monte Carlo samplings of Maxwell-Boltzmann distribution.
Once this is done the LGM immediately gives the cluster distribution using the rule that two nucleons are part of the same cluster if $\frac{r^2}{2\mu} - \epsilon_{i,n} - \epsilon_{j,n} < 0$. This method is similar to the Coniglio-Klein’s prescription [33] in condensed matter physics and was shown to be valid in LGM [26, 28, 30, 31]. To calculate clusters using MD we propagate the particles from the initial configuration for a long time under the influence of the chosen force. The form of the force is chosen to be also isospin dependent in order to compare with the results of LGM. The potential for unlike-nucleons is

$$v_{np}(r)\left(\frac{r}{r_0} < a\right) = C \left[ B(\frac{r_0}{r})^p - (\frac{r_0}{r})^q \right] e^{-\frac{1}{m^2 - r^2}},$$

$$v_{np}(r)\left(\frac{r}{r_0} > a\right) = 0.$$ ---- (2)

where $r_0 = 1.842 fm$ is the distance between the centers of two adjacent cubes. The parameters of the potentials are $p = 2, q = 1, a = 1.3, B = 0.924$, and $C = 1966$ MeV. With these parameters the potential is minimum at $r_0$ with the value $-5.33$ MeV, is zero when the nucleons are more than $1.32\sigma$ apart and becomes strongly repulsive when $r$ is significantly less than $r_0$. The potential for like-nucleons is written as

$$v_{pp}(r)(r < r_0) = v_{np}(r) - v_{np}(r_0),$$

$$v_{pp}(r)(r > r_0) = 0.$$ ---- (3)

The system evolves with the above potential. At asymptotic times the clusters are easily recognized. Observables based on the cluster distribution in the both models can now be compared. In the case of proton-proton interactions, the Coulomb interaction can also be added separately and it can be compared with the case without Coulomb effects.

In the simulation part of this report we choose the medium size nuclei $^{129}$Xe as an example. In most cases, $\rho_f$ is chosen to be $0.38 \rho_0$, since the experimental data can be best fitted by $\rho_f$ between $0.3 \rho_0$ and $0.4 \rho_0$ in previous LGM calculations [26, 34], which corresponds to $7^3$ cubic lattice.

In order to check the phase transition behavior in the L-LGM, we will firstly show the results of some physical observables, namely the effective power-law parameter, $\tau$, the second moment of the cluster distribution, $S_2$ [35], and the multiplicity of intermediate mass fragments, $N_{imf}$ for the disassembly of $^{129}$Xe in figure 1. These observables have been evidenced useful in previous works to judge the liquid gas phase transition, as shown in Ref. [27, 29, 36]. The valley of $\tau$, the peaks of $N_{imf}$ and $S_2$ happens around $T \sim 5.5$ MeV which is the signature of onset of phase transition. The above phase transition temperature will be only used as a reference of the novel signature, as stated below.

Now we present the results for testing Zipf’s law in the charge distribution of clusters. The law states that the relation between the sizes and their ranks is described by $Z_n = c/n$ (n=1, 2, 3, ...), where $c$ is a constant and $Z_n$ is the average charge (or mass) of rank $n$ in a charge (or mass) list when we arrange the clusters in the order of decreasing size. For instance the charge $Z_2$ of the second largest cluster with rank $n = 2$ is one-half of the charge $Z_1$ of the largest cluster, the charge $Z_3$ of the third largest cluster with rank $n = 3$ is one-third of the charge $Z_1$ of the largest cluster, and so on. In the simulations of this work, we averaged the charges for each rank in charge lists of the events: we averaged the charges for the largest clusters in each event, averaged them for the second largest clusters, averaged them for the third largest clusters, and so on. From the charges averaged, we examined the relation between the charges $Z_n$ and their ranks $n$. Figure 2 shows such relations of $Z_n$ and $n$ for Xe in different temperature. The histogram is the simulated results and the straight lines represent the fit with $Z_n \propto n^{-\lambda}$ in the range of $1 \leq n \leq 10$, where $\lambda$ is the slope parameter. $\lambda$ is $5.77$ at $T = 3$ MeV. Then we increased the temperature and examined the same relation and obtained $\lambda = 3.65$ and $1.53$ at $T = 4$ and $5$ MeV, respectively. Up to $T = 5.5$ MeV, $\lambda = 1.00$, i.e., at this temperature the relation is satisfied to the Zipf’s law: $Z_n \propto n^{-\lambda}$. When temperature continues to increases, $\lambda$ continues to decreases, for instance, $\lambda = 0.80$ at $T = 6$ MeV and $\lambda = 0.56$ at $T = 7$. This temperature having the Zipf’s law, denoted as $T_A$, is consistent with the phase transition temperature obtained in Fig. 1, illustrating that the Zipf’s law is also a good judgement to phase transition. From the statistical point of view, the Zipf’s law is related to the critical phenomenon [2, 37]. Figure 3a summarizes the parameter $\lambda$ as a function of temperature. Clearly the Zipf’s law ($\lambda = 1$) reveals at phase transition point.

In order to further illustrate that the Zipf’s law exists most probably in phase transition point, we directly reproduce the histograms with Zipf’s law: $Z_n = c/n$. In this case, $c$ is sole parameter, but what we are interested in is its truth of the hypothesis of Zipf’s law: the $\chi^2$ test. Figure 3b demonstrates the $\chi^2/ndf$ for the $Z_n - n$ relations at different $T$. As expected, there is the minimum $\chi^2/ndf$ around the phase transition temperature, which further support that Zipf’s law of the fragment distribution reveals when the liquid gas phase transition occurs.

Beside $0.38\rho_0$ was used for $^{129}$Xe, 0.18$\rho_0$, corresponding to $9^3$ cubic lattice and 0.60$\rho_0$, corresponding to $6^3$ cubic lattice of $\rho_f$ are also taken to compare and check the results with different $\rho_f$ values in the LGM case. It was found that the Zipf’s law is also valid when the liquid gas phase transition occurs at different freeze-out density. Moreover, we also investigated larger systems, such as $A = 274, 500$ and $830$, in the LGM case to see if the system behaves as expected in $^{129}$Xe. The results show that the Zipf’s law behavior still remains at the same phase transition temperature as the one extracted from the extreme values of $\tau$, $N_{imf}$ and $S_2$. It illustrates that nuclear Zipf’s law is suitable as a signal of phase transition in larger $A$ limit.
III. EXPERIMENTAL EVIDENCE

A. 4π data of Heavy Ion Collision

Using the TAMU NIMROD (Neutron Ion Multidetector for Reaction Oriented Dynamics) and beams from the TAMU K500 super-conducting cyclotron, we have probed the properties of excited projectile-like fragments produced in the reactions of 47 MeV/nucleon $^{40}$Ar + $^{27}$Al, $^{48}$Ti and $^{58}$Ni. Earlier work on the reaction mechanisms of near symmetric collisions of nuclei in the $20 < A < 64$ mass region at energies near the Fermi energy have demonstrated the essential binary nature of such collisions, even at relatively small impact parameters [38]. As a result, they prove to be very useful in preparing highly excited light nuclei with kinematic properties which greatly simplify the detection and identification of the products of their subsequent de-excitation [39].

The charged particle detector array of NIMROD, which is set inside a neutron ball, includes 166 individual CsI detectors arranged in 12 rings in polar angles from $\sim 3^\circ$ to $\sim 170^\circ$. Eight forward rings have the same geometrical design as the INDRA detector, but have less granularity. In these experiments each forward ring included two super-telescopes (composed of two Si-Si-CsI detectors) and three Si-CsI telescopes to identify intermediate mass fragments. The CsI detectors are Tl doped crystals read by photo-multiplier tubes. A pulse shape discrimination method using different responses of fast and slow components of the light output of the CsI crystals is employed to identify particles. In the CsI detectors Hydrogen and Helium isotopes were clearly identified
and Li fragments are also isolated from the heavier fragments. In the super-telescopes, all isotopes with atomic number \( Z \leq 8 \) were clearly identified and in all telescopes particles were identified in atomic number. The NIMROD neutron ball, which surrounds the charged particle array, was used to determine the neutron multiplicities for selected events. The neutron ball consists of two hemispherical end caps and a central cylindrical section. The hemispheres are upstream and downstream of the charged particle array. They are 150 cm in diameter with beam pipe holes in the center. The central cylindrical sections 1.25m long with an inner hole of 60 cm diameter and 150 cm outer diameter. It is divided into four segments in the azimuthal angle direction. Between the hemispheres and the central section, there are 20 cm air gaps for cables and a duct for a pumping station. The neutron ball is filled with a pseudocumene based liquid scintillator mixed with 0.3 weight percent of Gd salt (Gd 2-ethyl hexanoate). Scintillation from a thermal neutron captured by Gd is detected by five 5-in phototubes in each hemisphere and three phototubes in each segment of the central section.

The correlation of the charged particle multiplicity \((M_{\text{qp}})\) and the neutron multiplicity \((M_{n})\) was used to sort event violence. Recently we have developed a new method for the assignment of each light charged particle (LCP) to an emission source [6]. This is done with a combination of three source fits and Monte-Carlo sampling techniques. We first obtain the laboratory energy spectra for different LCP at different laboratory angles and reproduce them using the three source fits. In the laboratory frame, the energy spectra of LCP can be modeled as the overlap of emission from three independent moving equilibrated sources, i.e. the QP, NN and QT sources. Intermediate mass fragments, IMF, with \( Z \geq 4 \) were identified in the telescope modules of NIMROD. For such ejectiles we have we have not used such fitting techniques. Rather we have used a rapidity cut \((> 0.65 \text{ beam rapidity})\) to assign IMF to the QP source.

To further explore this region we have investigated other proposed observables commonly related to fluctuations and critical behavior. Fig. 4(c) shows the mean normalized second moment \([35], \langle S_2 \rangle\) as a function of excitation energy. A peak is seen around 5.6 MeV/u, it indicates that the fluctuation of the fragment distribution is the largest in this excitation energy region. Similarly, the normalized variance in \( Z_{\text{max}}/Z_{\text{QP}} \) distribution (i.e. \( \text{NVZ} = \frac{\sigma_{Z_{\text{max}}/Z_{\text{QP}}}}{Z_{\text{max}}/Z_{\text{QP}}} \)) [41] shows a maximum in the same excitation energy region [Fig. 4(d)], which illustrates the maximal fluctuation for the largest fragment (order parameter) is reached around \( E^*/A = 5.6 \text{ MeV} \). Except the largest fragment, the second largest fragment also shows its importance in the above turning point. Fig. 4(e) shows a broad peak of \( \langle Z_{\text{max}} \rangle \) - the average atomic number of the second largest fragment exists at 5.6 MeV/u.

The significance of the 5-6 MeV region in our data is further indicated by a Zipf’s law analysis. In Fig. 5 we present Zipf plots for rank ordered average \( Z \) in the nine different energy bins. Lines in the figure are fits to the power law expression \( \langle Z_n \rangle \propto n^{-\lambda} \). Fig. 4(f) shows the fitted \( \lambda \) parameter as a function of excitation energy. As shown in Fig. 5, this rank ordering of the probability observation of fragments of a given atomic number, from largest to the smallest, does indeed lead to a Zipf’s power law parameter \( \lambda = 1 \) in the 5-6 MeV/nucleon range. When \( \lambda \sim 1 \), Zipf’s law is satisfied. In this case, the mean size of the second largest fragment is 1/2 of that of the the largest fragment; That of the third largest fragment is 1/3 of the largest fragment, etc.

## B. CERN Emulsion Experiment

The proposed nuclear Zipf-type plot has been also applied in the analysis of CERN emulsion or Plastic data of Pb + Pb or Plastic at 158 AGeV and it was found that the nuclear Zipf law is satisfied when the liquid gas phase transition occurs [42, 43].

Dabrowska et al. extends the studies to the multifragmentation of lead projectiles at an energy of 158 AGeV [42]. The analyzed data were obtained from the CERN EMU13 experiment in which emulsion chambers, composed of nuclear target foils and thin emulsion plates interleaved with spacers, allow for precise measurements of emission angles and charges of all projectile fragments.
FIG. 4: The effective Fisher-law parameter ($\tau_{eff}$) (a), the effective exponential law parameter ($\lambda_{eff}$) (b), $\langle S_2 \rangle$ (c), NVZ fluctuation (d), the mean charge number of the second largest fragment ($\langle Z_{2max} \rangle$) (e), the Zipf-law parameter $\lambda$ (f). See details in text.

FIG. 5: Zipf plots in nine different excitation energy bins for the QP formed in $^{40}\text{Ar} + ^{58}\text{Ni}$. The dots are data and the lines are Zipf-law fits. The statistical error is smaller than the size of the circles.

FIG. 6: Zipf law fit to the dependences of the mean charge of the fragment on its rank. The different symbols represent the multifragmentation data of different beam with emulsion target. Circles and solid line represent Pb beam at 158 AGeV, squares and dashed line represents Au beam at 10.6 AGeV, star and dotted line represent Au beam at 0.64 AGeV. The data is taken from Ref. [42].

emitted from Pb-Nucleus interactions. The results on fragment multiplicities, charge distributions and angular correlations are analyzed for multifragmentation of the Pb projectile after an interaction with heavy (Pb) and light (Plastic - $C_5H_4O_2$) targets. Detailed description of the emulsion experiment can be found in Ref. [42].

Fig. 6 shows the Zipf-type plot for the charged fragments heavier than helium emitted in multifragmentation process of Au or Pb projectile at different primary energies. The values of $\lambda$ exponents from fits ($Z_n \sim n^{-\lambda}$) are $0.92 \pm 0.03$, $0.90 \pm 0.02$ and $0.96 \pm 0.04$ for primary energies of 158, 10.6 and 0.64 AGeV, respectively. Within the statistical errors the values of the $\lambda$ coefficient are the same in the studied energy interval ($< 1$-158) AGeV and do not differ significantly from unity [42].

Dabrowska et al. also studied the dependence of the power law exponent $\lambda$ on the control parameter $m$, the normalized multiplicities by the total charge of spectator particles [43]. In Fig. 7(a) the mean multiplicity ($N_f$) of fragments with $Z \geq 3$ and the mean number ($N_{IMF}$) of the intermediate fragments. The latter are usually defined as fragments with $3 \leq Z \leq 30$. In Fig. 7(b) the dependence of the exponent $\tau$ of the power fits to the charge distribution of fragments, perfomed at different ranges of $m$. In this analysis the fits are restricted to fragment charges smaller than $Z = 16$. At small values of $m$ a system has few light fragments and the power law is steep; at large values of $m$ there are many light
FIG. 7: (a) Mean number, \( \langle N_f \rangle \), of fragments (squares) and mean number, \( \langle N_{IMF} \rangle \), of intermediate mass fragments (circles) as a function of the normalized multiplicity \( m \). Error bars are smaller than the size of the squares and circles. (b) Power law exponent, \( \tau \), of the charge distribution of fragments in different intervals of \( m \). (c) Power law exponent, \( \lambda \), in the Zipf’s law (see text) in different intervals of \( m \). Error bars are smaller then data points. The data is taken from Ref. [43].

fragments and little else leading again a steep power law. At the moderate excitation energies where heavier fragments appear and where we expect the phase transition, the exponent \( \tau \) has its lowest value. As can be seen from Fig. 7(b) it happens for \( m \) values between 0.35 and 0.55. In Fig. 7(c) the dependence of \( \lambda \) obtained from the fits \( \langle Z_n \rangle \sim n^{-\lambda} \), on \( m \) is depicted. The exponent \( \lambda \) decreases with increasing \( m \) from about 0.3 to 0.5 the value of \( \lambda \) is close to unity and the Zipf’s law is satisfied. This suggest that at this value of \( m \) the liquid gas phase transition occurs. It has been checked that \( \lambda = 1 \) occurs in the same region of \( m \) irrespectively of the mass of the target [43]. This means that the liquid gas phase transition occurs when a given amount of energy is deposited into the nucleus and does not depend on the mass of the target. As expected for liquid gas phase transition, the previously shown maxima in frequency distributions of multiply charged fragments (Fig. 7(a)) as well as a minimum of the power law parameter \( \tau \) (Fig. 7(b)), all occur at the same values of \( m \), where Zipf’s law is satisfied.

IV. CONCLUSION AND OUTLOOK

In conclusion, nuclear Zipf’s law has been proposed and it has been used to study the liquid gas phase transition of nuclei. At the point of phase transition, the cluster mass (charge) shows exactly inversely to its rank, i.e. Zipf’s law appears. Even though the criterion is still phenomenological, it is simple and practicable tools to diagnose the nuclear liquid gas phase transition in experiments and theories. The \( 4\pi \) data of heavy ion collision for quasi-projectile fragmentation of Ar and the CERN emulsion/plastic data have confirmed that the nuclear Zipf law is a valid probe for the liquid gas phase transition. The satisfaction of the Zipf law for the cluster distributions around the liquid gas phase transition illustrates that the clusters obey a particular rank ordering distribution rather than the equal-size fragment distribution as the spinodal instability predicts [46–50]. It looks useful to distinguish the mechanism of the phase transition from the nuclear Zipf-type plots.

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